# CHAPTER 4 <br> LAGRANGIAN MESHES 

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### 4.1 INTRODUCTION

In Lagrangian meshes, the nodes and elements move with the material. Boundaries and interfaces remain coincident with element edges, so that their treatment is simplified. Quadrature points also move with the material, so constitutive equations are always evaluated at the same material points, which is advantageous for history dependent materials. For these reasons, Lagrangian meshes are widely used for solid mechanics.

The formulations described in this Chapter apply to large deformations and nonlinear materials, i.e. they consider both geometric and material nonlinearities. They are only limited by the element's capabilities to deal with large distortions. The limited distortions most elements can sustain without degradation in performance or failure is an important factor in nonlinear analysis with Lagrangian meshes and is considered for several elements in the examples.

Finite element discretizations with Lagrangian meshes are commonly classified as updated Lagrangian formulations and total Lagrangian formulations. Both formulations use Lagrangian descriptions, i.e. the dependent variables are functions of the material (Lagrangian) coordinates and time. In the updated Lagrangian formulation, the derivatives are with respect to the spatial (Eulerian) coordinates; the weak form involves integrals over the deformed (or current) configuration. In the total Lagrangian formulation, the weak form involves integrals over the initial (reference ) configuration and derivatives are taken with respect to the material coordinates.

This Chapter begins with the development of the updated Lagrangian formulation. The key equation to be discretized is the momentum equation, which is expressed in terms of the Eulerian (spatial) coordinates and the Cauchy (physical) stress. A weak form for the momentum equation is then developed, which is known as the principle of virtual power. The momentum equation in the updated Lagrangian formulation employs derivatives with respect to the spatial coordinates, so it is natural that the weak form involves integrals taken with respect to the spatial coordinates, i.e. on the current configuration. It is common practice to use the rate-of-deformation as a measure of strain rate, but other measures of strain or strain-rate can be used in an updated Lagrangian formulation. For many applications, the updated Lagrangian formulation provides the most efficient formulation.

The total Lagrangian formulation is developed next. In the total Lagrangian formulation, we will use the nominal stress, although the second Piola-Kirchhoff stress is also used in the formulations presented here. As a measure of strain we will use the Green strain tensor in the total Lagrangian formulation. A weak form of the momentum equation is developed, which is known as the principle of virtual work. The development of the toal Lagrangian formulation closely parallels the updated Lagrangian formulation, and it is stressed that the two are basically identical. Any of the expressions in the updated Lagrangian formulation can be transformed to the total Lagrangian formulation by transformations of tensors and mappings of configurations. However, the total Lagrangian formulation is often used in practice, so to understand the literature, an
advanced student must be familiar with it. In introductory courses one of the formulations can be skipped.

Implementations of the updated and total Lagrangian formulations are given for several elements. In this Chapter, only the expressions for the nodal forces are developed. It is stressed that the nodal forces represent the discretization of the momentum equation. The tangential stiffness matrices, which are emphasized in many texts, are simply a means to solving the equations for certain solution procedures. They are not central to the finite element discretization. Stiffness matrices are developed in Chapter 6.

For the total Lagrangian formulation, a variational principle is presented. This principle is only applicable to static problems with conservative loads and hyperelastic materials, i.e. materials which are described by a path-independent, rate-independent elastic constitutive law. The variational principle is of value in interpreting and understanding numerical solutions and the stability of nonlinear solutions. It can also sometimes be used to develop numerical procedures.

### 4.2 GOVERNING EQUATIONS

We consider a body which occupies a domain $\Omega$ with a boundary $\Gamma$. The governing equations for the mechanical behavior of a continuous body are:

1. conservation of mass (or matter)
2. conservation of linear momentum and angular momentum
3. conservation of energy, often called the first law of thermodynamics
4. constitutive equations
5. strain-displacement equations


Figure 4.0. Deformed and undeformed body showing a set of admissible lines of interwoven discontinuities $\Gamma^{\text {int }}$ and the notation.

We will first develop the updated Lagrangian formulation. The conservation equations have been developed in Chapter 3 and are given in both tensor form and indicial form in Box 4.1. As can be
seen, the dependent variables in the conservation equations are written in terms of material coordinates but are expressed in terms of what are classically Eulerian variables, such as the Cauchy stress and the rate-of-deformation.

We next give a count of the number of equations and unknowns. The conservation of mass and conservation of energy equations are scalar equations. The equation for the conservation of linear momentum, or momentum equation for short, is a tensor equation which consists of $n_{S D}$ partial differential equations, where $n_{S D}$ is the number of space dimensions. The constitutive equation relates the stress to the strain or strain-rate measure. Both the strain measure and the stress are symmetric tensors, so this provides $n_{\sigma}$ equations where

$$
\begin{equation*}
n_{\sigma} \equiv n_{S D}\left(n_{S D}+1\right) / 2 \tag{4.2.1}
\end{equation*}
$$

In addition, we have the $n_{\sigma}$ equations which express the rate-of-deformation $\mathbf{D}$ in terms of the velocities or displacements. Thus we have a total of $2 n_{\sigma}+n_{S D}+1$ equations and unknowns. For example, in two-dimensional problems ( $n_{S D}=2$ ) without energy transfer, so we have nine partial differential equations in nine unknowns: the two momentum equations, the three constitutive equations, the three equations relating $\mathbf{D}$ to the velocity and the mass conservation equation. The unknowns are the three stress components (we assume symmetry of the stress), the three components of $\mathbf{D}$, the two velocity components, and the density $\rho$, for a total of 9 unknowns. Additional unknown stresses (plane strain) and strains (plane stress) are evaluated using the plane strain and plane stress conditions, respectively. In three dimensions ( $n_{S D}=3, n_{\sigma}=6$ ), we have 16 equations in 16 unknowns.

When a process is neither adiabatic nor isothermal, the energy equation must be appended to the system. This adds one equation and $n_{S D}$ unknowns, the heat flux vector $q_{i}$. However, the heat flux vector can be determined from a single scalar, the temperature, so only one unknown is added; the heat flux is related to the temperature by a type of constitutive law which depends on the material. Usually a simple linear relation, Fourier's law, is used. This then completes the system of equations, although often a law is needed for conversion of some of the mechanical energy to thermal energy; this is discussed in detail in Section 4.10.

The dependent variables are the velocity $\mathbf{v}(\mathbf{X}, t)$, the Cauchy stress $\sigma(\mathbf{X}, t)$, the rate-ofdeformation $\mathbf{D}(\mathbf{X}, t)$ and the density $\rho(\mathbf{X}, t)$. As seen from the preceding a Lagrangian description is used: the dependent variables are functions of the material (Lagrangian) coordinates. The expression of all functions in terms of material coordinates is intrinsic in any treatment by a Lagrangian mesh. In principle, the functions can be expressed in terms of the spatial coordinates at any time $t$ by using the inverse of the map $\mathbf{x}=\phi(\mathbf{X}, t)$. However, inverting this map is quite difficult. In the formulation, we shall see that it is only necessary to obtain derivatives with respect to the spatial coordinates. This is accomplished by implicit differentiation, so the map corresponding to the motion is never explicitly inverted.

In Lagrangian meshes, the mass conservation equation is used in its integrated form (B4.1.1) rather than as a partial diffrential equation. This eliminates the need to treat the continuity equation, (3.5.20). Although the continuity equation can be used to obtain the density in a Lagrangian mesh, it is simpler and more accurate to use the integrated form (B4.1.1)

The constitutive equation (Eq. B4.1.5), when expressed in rate form in terms of a rate of Cauchy stress, requires a frame invariant rate. For this purpose, any of the frame-invariant rates,
such as the Jaumann or the Truesdell rate, can be used as described in Chapter 3. It is not necessary for the constitutive equation in the updated Lagrangian formulation to be expressed in terms of the Cauchy stress or its frame invariant rate. It is also possible to use constitutive equations expressed in terms of the PK2 stress and then to convert the PK2 stress to a Cauchy stress using the transformations developed in Chapter 3 prior to computing the internal forces.

The rate-of-deformation is used as the measure of strain rate in Eq. (B4.1.5). However, other measures of strain or strain-rate can also be used in an updated Lagrangian formulation. For example, the Green strain can be used in updated Lagrangian formulations. As indicated in Chapter 3, simple hypoelastic laws in terms of the rate-of-deformation can cause difficulties in the simulation of cyclic loading because its integral is not path independent. However, for many simulations, such as the single application of a large load, the errors due to the path-dependence of the integral of the rate-of-deformation are insignificant compared to other sources of error, such as inaccuracies and uncertainties in the material data and material model. The appropriate selection of stress and strain measures depends on the constitutive equation, i.e. whether the material response is reversible or not, time dependence, and the load history under consideration.

The boundary conditions are summarized in Eq. (B4.1.7). In two dimensional problems, each component of the traction or velocity must be prescribed on the entire boundary; however the same component of the traction and velocity cannot not be prescribed on any point of the boundary as indicated by Eq. (B.4.1.8). Traction and velocity components can also be specified in local coordinate systems which differ from the global system. An identical rule holds: the same components of traction and velocity cannot be prescribed on any point of the boundary. A velocity boundary condition is equivalent to a displacement boundary condition: if a displacement is specified as a function of time, then the prescribed velocity can be obtained by time differentiation; if a velocity is specified, then the displacement can be obtained by time integration. Thus a velocity boundary condition will sometimes be called a displacement boundary condition, or vice versa.

The initial conditions can be applied either to the velocities and the stresses or to the displacements and velocities. The first set of initial conditions are more suitable for most engineering problems, since it is usually difficult to determine the initial displacement of a body. On the other hand, initial stresses, often known as residual stresses, can sometimes be measured or estimated by equilibrium solutions. For example, it is almost impossible to determine the displacements of a steel part after it has been formed from an ingot. On the other hand, good estimates of the residual stress field in the engineering component can often be made. Similarly, in a buried tunnel, the notion of initial displacements of the soil or rock enclosing the tunnel is quite meaningless, whereas the initial stress field can be estimated by equilibrium analysis. Therefore, initial conditions in terms of the stresses are more useful.

BOX 4.1
Governing Equations for Updated Lagrangian Formulation
conservation of mass

$$
\begin{equation*}
\rho(\mathbf{X}) J(\mathbf{X})=\rho_{0}(\mathbf{X}) J_{0}(\mathbf{X})=\rho_{0}(\mathbf{X}) \tag{B4.1.1}
\end{equation*}
$$

conservation of linear momentum

$$
\begin{equation*}
\nabla \cdot \sigma+\rho \mathbf{b}=\rho \dot{\mathbf{v}} \equiv \rho \frac{D \mathbf{v}}{D t} \quad \text { or } \quad \frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i}=\rho \dot{v}_{i} \equiv \rho \frac{D v_{i}}{D t} \tag{B4.1.2}
\end{equation*}
$$

conservation of angular momentum: $\sigma=\sigma^{T}$ or $\sigma_{i j}=\sigma_{j i}$
conservation of energy:

$$
\begin{equation*}
\rho \dot{w}^{i n t}=\mathbf{D}: \sigma-\nabla \cdot \mathbf{q}+\rho s \quad \text { or } \quad \rho \dot{w}^{i n t}=D_{i j} \sigma_{j i}-\frac{\partial q_{i}}{\partial x_{i}}+\rho s \tag{B4.1.4}
\end{equation*}
$$

constitutive equation: $\sigma^{\nabla}=S_{t}^{\sigma D}(\mathbf{D}, \sigma$, etc. $)$
rate-of-deformation: $\mathbf{D}=\operatorname{sym}(\nabla \mathbf{v}) \quad D_{i j}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right)$
boundary conditions

$$
\begin{array}{llll}
n_{j} \sigma_{j i}=\bar{t}_{i} \text { on } & \Gamma_{t_{i}} \quad \mathrm{v}_{i}=\overline{\mathrm{v}}_{i} \quad \text { on } & \Gamma_{v_{i}} \\
\Gamma_{t_{i}} \cap \Gamma_{v_{i}}=0 & & \Gamma_{t_{i}} \cup \Gamma_{v_{i}}=\Gamma &  \tag{B4.1.8}\\
i=1 \text { to } n_{S D}
\end{array}
$$

initial conditions

$$
\begin{array}{ll}
\mathbf{v}(\mathbf{x}, 0)=\mathbf{v}_{0}(\mathbf{x}) & \sigma(\mathbf{x}, 0)=\sigma_{0}(\mathbf{x}) \\
\text { or } & \\
\mathbf{v}(\mathbf{x}, 0)=\mathbf{v}_{0}(\mathbf{x}) & \mathbf{u}(\mathbf{x}, 0)=\mathbf{u}_{0}(\mathbf{x}) \tag{B4.1.10}
\end{array}
$$

interior continuity conditions (stationary)

$$
\begin{equation*}
\text { on } \Gamma_{i n t}:\langle\mathbf{n} \cdot \sigma\rangle=0 \quad \text { or } \quad\left\langle n_{i} \sigma_{i j}\right\rangle \equiv n_{i}^{A} \sigma_{i j}^{A}+n_{i}^{B} \sigma_{i j}^{B}=0 \tag{B4.1.11}
\end{equation*}
$$

We have also included the interior continuity conditions on the stresses in Box 4.1as Eq. (B4.1.11). In this equation, superscripts $A$ and $B$ refer to the stresses and normal on two sides of the discontinuity: see Section 3.5.10. These continuity conditions must be met by the tractions wherever stationary discontinuites in certain stress and strain components are possible, such as at material interfaces. They must hold for bodies in equilibrium and in transient problems. As mentioned in Chapter 2, in transient problems, moving discontinuities are also possible; however, moving discontinuities are treated in Lagrangian meshes by smearing them over several elements. Thus the moving discontinuity conditions need not be explicitly stated. Only the stationary continuity conditions are imposed explicitly by a finite element approximation.

### 4.3 WEAK FORM: PRINCIPLE OF VIRTUAL POWER

In this section, the principle of virtual power, is developed for the updated Lagrangian formulation. The principle of virtual power is the weak form for the momentum equation, the traction boundary conditions and the interior traction continuity conditions. These three are collectively called generalized momentum balance. The relationship of the principle of virtual power to the momentum equations will be described in two parts:

1. The principle of virtual power (weak form) will be developed from the generalized momentum balance (strong form), i.e. strong form to weak form.
2. The principle of virtual power (weak form) will be shown to imply the generalized momentum balance (strong form), i.e. weak form to strong form.

We first define the spaces for the test functions and trial functions. We will consider the minimum smoothness required for the functions to be defined in the sense of distributions, i.e. we allow Dirac delta functions to be derivatives of functions. Thus, the derivatives will not be defined
according to classical definitions of derivatives; instead, we will admit derivatives of piecewise continuous functions, where the derivatives include Dirac delta functions; this generalization was discussed in Chapter 2.

The space of test functions is defined by:

$$
\begin{equation*}
\delta v_{j}(\mathbf{X}) \in \mathcal{U}_{0} \quad \mathcal{U}_{0}=\left\{\delta v_{i} \mid \delta v_{i} \in C^{0}(\mathbf{X}), \delta v_{i}=0 \text { on } \Gamma_{v_{i}}\right\} \tag{4.3.1}
\end{equation*}
$$

This selection of the space for the test functions $\delta \mathbf{v}$ is dictated by foresight from what will ensue in the development of the weak form; with this construction, only prescribed tractions are left in the final expression of the weak form. The test functions $\delta \mathbf{v}$ are sometimes called the virtual velocities.

The velocity trial functions live in the space given by

$$
\begin{equation*}
v_{i}(\mathbf{X}, t) \in \mathcal{U} \quad \mathcal{U}=\left\{v_{i} \mid v_{i} \in C^{0}(\mathbf{X}), v_{i}=\bar{v}_{i} \text { on } \Gamma_{v_{i}}\right\} \tag{4.3.2}
\end{equation*}
$$

The space of displacements in $\mathcal{U}$ is often called kinematically admissible displacements or compatible displacements; they satisfy the continuity conditions required for compatibility and the velocity boundary conditions. Note that the space of test functions is identical to the space of trial functions except that the virtual velocities vanish wherever the trial velocities are prescribed. We have selected a specific class of test and trial spaces that are applicable to finite elements; the weak form holds also for more general spaces, which is the space of functions with square integrable derivatives, called a Hilbert space.

Since the displacement $u_{i}(\mathbf{X}, t)$ is the time integral of the velocity, the displacement field can also be considered to be the trial function. We shall see that the constitutive equation can be expressed in terms of the displacements or velocities. Whether the displacements or velocities are considered the trial functions is a matter of taste.
4.3.1 Strong Form to Weak Form. As we have already noted, the strong form, or generalized momentum balance, consists of the momentum equation, the traction boundary conditions and the traction continuity conditions, which are respectively:

$$
\begin{align*}
& \frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i}=\rho \dot{v}_{i} \text { in } \Omega  \tag{4.3.3a}\\
& n_{j} \sigma_{j i}=\bar{t}_{i} \text { on } \Gamma_{t_{i}}  \tag{4.3.3b}\\
& \left\{n_{j} \sigma_{j i}\right\}=0 \quad \text { on } \Gamma_{i n t} \tag{4.3.3c}
\end{align*}
$$

where $\Gamma_{i n t}$ is the union of all surfaces (lines in two dimensions) on which the stresses are discontinuous in the body.

Since the velocities are $C^{0}(\mathbf{X})$, the displacements are similarly $C^{0}(\mathbf{X})$; the rate-ofdeformation and the rate of Green strain will then be $C^{-1}(\mathbf{X})$ since they are related to spatial derivatives of the velocity. The stress $\sigma$ is a function of the velocities via the constitutive equation (B4.1.4relates the rate-of-deformation to the velocities) and Eq. (B4.1.5), which or the Green
strain to the displacement. It is assumed that the constitutive equation leads to a stress that is a well-behaved function of the Green strain tensor, so that the stresses will also be $C^{-1}(\mathbf{X})$. Note that the stress rate is often not a continuous function of the rate-of-deformation; for example, it is discontinuous at the transition between plastic behavior and elastic unloading.

The first step in the development of the weak form, as in the one-dimensional case in Chapter 2, consists of taking the product of a test function $\delta v_{i}$ with the momentum equation and integrating over the current configuration:

$$
\begin{equation*}
\int_{\Omega} \delta v_{i}\left(\frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i}-\rho \dot{v}_{i}\right) d \Omega=0 \tag{4.3.4}
\end{equation*}
$$

In the intergral, all variables must be implicitly transformed to be functions of the Eulerian coordinates by (???). However, this transformation is never needed in the implementation. The first term in (4.3.4) is next expanded by the product rule, which gives

$$
\begin{equation*}
\int_{\Omega} \delta v_{i} \frac{\partial \sigma_{j i}}{\partial x_{j}} d \Omega=\int_{\Omega}\left[\frac{\partial}{\partial x_{j}}\left(\delta v_{i} \sigma_{j i}\right)-\frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{j i}\right] d \Omega \tag{4.3.5}
\end{equation*}
$$

Since the velocities are $C^{0}$ and the stresses are $C^{-1}$, the term $\delta v_{i} \sigma_{j i}$ on the RHS of the above is $C^{-1}$. We assume that the discontinuities occur over a finite set of surfaces $\Gamma_{i n t}$, so Gauss's theorem, Eq. (3.5.4) gives

$$
\begin{equation*}
\left.\int_{\Omega} \frac{\partial}{\partial x_{j}}\left(\delta v_{i} \sigma_{j i}\right) d \Omega=\int_{\Gamma_{\text {int }}} \delta v_{i}{ }_{l} n_{j} \sigma_{j i}\right\rangle d \Gamma+\int_{\Gamma} \delta v_{i} n_{j} \sigma_{j i} d \Gamma \tag{4.3.6}
\end{equation*}
$$

From the strong form (4.3.3c), the first integral on the RHS vanishes. For the second integral on the RHS we can use another part of the strong form, the traction boundary conditions (4.3.3b) on the prescribed traction boundaries. Since the test function vanishes on the complement of the traction boundaries, (4.3.6) gives

$$
\begin{equation*}
\int_{\Omega} \frac{\partial}{\partial x_{j}}\left(\delta v_{i} \sigma_{j i}\right) d \Omega=\sum_{i=1}^{n_{S D}} \int_{\Gamma_{\Gamma_{i}}} \delta v_{i} \bar{t}_{i} d \Gamma \tag{4.3.7}
\end{equation*}
$$

The summation sign is included on the RHS to avoid any confusion arising from the presence of a third index $i$ in $\Gamma_{t i}$; if this index is ignored in the summation convention then there is no need for a summation sign.

If (4.3.7) is substituted into (4.3.4) we obtain

$$
\begin{equation*}
\int_{\Omega} \delta v_{i} \frac{\partial \sigma_{j i}}{\partial x_{j}} d \Omega=\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t_{i}}} \delta v_{i} \bar{t}_{i} d \Gamma-\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{j i} d \Omega \tag{4.3.8}
\end{equation*}
$$

The process of obtaining the above is called integration by parts. If Eq. (4.3.8) is then substituted into (4.3.4), we obtain

$$
\begin{equation*}
\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{j i} d \Omega-\int_{\Omega} \delta v_{i} \rho b_{i} d \Omega-\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t_{i}}} \delta v_{i} \bar{t}_{i} d \Gamma+\int_{\Omega} \delta v_{i} \rho \dot{v}_{i} d \Omega=0 \tag{4.3.9}
\end{equation*}
$$

The above is the weak form for the momentum equation, the traction boundary conditions and the interior continuity conditions. It is known as the principle of virtual power, see Malvern (1969), for each of the terms in the weak form is a virtual power; see Section 2.5.
4.3.2. Weak Form to Strong Form. It will now be shown that the weak form (4.3.9) implies the strong form or generalized momentum balance: the momentum equation, the traction boundary conditions and the interior continuity conditions, Eqs. (4.3.3). To obtain the strong form, the derivative of the test function must be eliminated from (4.3.9). This is accomplished by using the derivative product rule on the first term, which gives

$$
\begin{equation*}
\int_{\Omega} \frac{\partial\left(\delta \mathrm{v}_{i}\right)}{\partial x_{j}} \sigma_{j i} d \Omega=\int_{\Omega} \frac{\partial\left(\delta \mathrm{v}_{i} \sigma_{j i}\right)}{\partial x_{j}} d \Omega-\int_{\Omega} \delta \mathrm{v}_{i} \frac{\partial \sigma_{j i}}{\partial x_{j}} d \Omega \tag{4.3.10}
\end{equation*}
$$

We now apply Gauss's theorem, see Section 3.5.2, to the first term on the RHS of the above

$$
\begin{align*}
& \int_{\Omega} \frac{\partial\left(\delta v_{i} \sigma_{j i}\right)}{\partial x_{j}} d \Omega=\int_{\Gamma} \delta v_{i} n_{j} \sigma_{j i} d \Gamma+\int_{\Gamma_{i n t}} \delta v_{i}\left\langle n_{j} \sigma_{j i}\right\rangle d \Gamma=  \tag{4.3.11}\\
& \left.\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t i}} \delta v_{i} n_{j} \sigma_{j i} d \Gamma+\int_{\Gamma_{i n t}} \delta v_{i} / n_{j} \sigma_{j i}\right\rangle d \Gamma
\end{align*}
$$

where the second equality follows because $\delta v_{i}=0$ on $\Gamma_{v_{i}}$, (see Eq. (4.3.1) and Eq. (B4.1.7)). Substituting Eq. (4.3.11) into Eq. (4.3.10) and in turn to (4.3.9), we obtain

$$
\begin{equation*}
\int_{\Omega} \delta v_{i}\left(\frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i}-\rho \dot{v_{i}}\right) d \Omega-\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t_{i}}} \delta v_{i}\left(n_{j} \sigma_{j i}-\bar{t}_{i}\right) d \Gamma-\int_{\Gamma_{i n t}} \delta v_{i} n^{\prime} n_{j} \sigma_{j i} \backslash d \Gamma=0 \tag{4.3.12}
\end{equation*}
$$

We will now prove that the coefficients of the test functions in the above integrals must vanish. For this purpose, we prove the following theorem

$$
\begin{align*}
& \text { if } \alpha_{i}(\mathbf{X}), \beta_{i}(\mathbf{X}), \gamma_{i}(\mathbf{X}) \in C^{-1} \text { and } \delta v_{i}(\mathbf{X}) \in \mathcal{U}_{0} \\
& \text { and } \int_{\Omega} \delta v_{i} \alpha_{i} d \Omega+\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t_{i}}} \delta v_{i} \beta_{i} d \Gamma+\int_{\Gamma_{\text {int }}} \delta v_{i} \gamma_{i} d \Gamma=0 \quad \forall \delta v_{i}(\mathbf{X})  \tag{4.3.13}\\
& \text { then } \alpha_{i}(\mathbf{X})=0 \text { in } \Omega, \beta_{i}(\mathbf{X})=0 \text { on } \Gamma_{t_{i}}, \gamma_{i}(\mathbf{X})=0 \text { on } \Gamma_{\text {int }}
\end{align*}
$$

where the integral is either transformed to the reference configuration or the variables are expressed in terms of the Eulerian coordinates by the inverse map prior to evaluation of the integrals.

In functional analysis, the statement in (4.3.13) is called the density theorem, Oden and Reddy (1976, p.19). It is also called the fundamental theorem of variational calculus; sometimes we call it the function scalar product theorem since it is the counterpart of the scalar product theorem given in Chapter 2. We follow Hughes [1987, p.80] in proving (4.3.13). As a first step we show that $\alpha_{i}(\mathbf{X})=0$ in $\Omega$. For this purpose, we assume that

$$
\begin{equation*}
\delta v_{i}(\mathbf{X})=\alpha_{i}(\mathbf{X}) f(\mathbf{X}) \tag{4.3.14}
\end{equation*}
$$

where

1. $f(\mathbf{X})>0$ on $\Omega$ but $f(\mathbf{X})=0$ on $\Gamma_{i n t}$ and $f(\mathbf{X})=0$ on $\Gamma_{t_{i}}$
2. $f(\mathbf{X})$ is $C^{-1}$

Substituting the above expression for $\delta v_{i}$ into (4.3.13) gives

$$
\begin{equation*}
\int_{\Omega} \alpha_{i}(\mathbf{X}) \alpha_{i}(\mathbf{X}) f(\mathbf{X}) d \Omega=0 \tag{4.3.15}
\end{equation*}
$$

The integrals over the boundary and interior surfaces of discontinuity vanish because the arbitrary function $f(\mathbf{X})$ has been chosen to vanish on these surfaces. Since $f(\mathbf{X})>0$, and the functions $f(\mathbf{X})$ and $\alpha_{i}(\mathbf{X})$ are sufficiently smooth, Equation (4.3.15) implies $\alpha_{i}(\mathbf{X})=0$ in $\Omega$ for $i=1$ to $n_{S D}$

To show that the $\gamma_{i}(\mathbf{X})=0$, let

$$
\begin{equation*}
\delta v_{i}(\mathbf{X})=\gamma_{i}(\mathbf{X}) f(\mathbf{X}) \tag{4.3.16}
\end{equation*}
$$

where

1. $f(\mathbf{x})>0$ on $\Gamma_{\text {int }} ; f(\mathbf{x})=0$ on $\Gamma_{t_{i}}$;
2. $f(\mathbf{x})$ is $C^{-1}$

Substituting (4.3.16) into (4.3.13) gives

$$
\begin{equation*}
\int_{\Gamma_{\text {iutt }}} \gamma_{i}(\mathbf{x}) \gamma_{i}(\mathbf{x}) f(\mathbf{x}) d \Gamma=0 \tag{4.3.17}
\end{equation*}
$$

which implies $\gamma_{i}(\mathbf{x})=0$ on $\Gamma_{i n t}($ since $f(\mathbf{x})>0)$.
The final step in the proof, showing that $\beta_{i}(\mathbf{x})=0$ is accomplished by using a function $f(\mathbf{x})>0$ on $\Gamma_{t_{i}}$. The steps are exactly as before. Thus each of the $\alpha_{i}(\mathbf{x}), \beta_{i}(\mathbf{x})$, and $\gamma_{i}(\mathbf{x})$ must vanish on the relevant domain or surface. Thus Eq. (4.3.12) implies the strong form: the momentum equation, the traction boundary conditions, and the interior continuity conditions, Eqs. (4.3.3).

Let us now recapitulate what has been accomplished so far in this Section. We first developed a weak form, called the principle of virtual power, from the strong form. The strong form consists of the momentum equation, the traction boundary conditions and jump conditions.

The weak form was obtained by multiplying the momentum equation by a test function and integrating over the current configuration. A key step in obtaining the weak form is the elimination of the derivatives of the stresses, Eq. (4.3.5-6). This step is crucial since as a result, the stresses can be $C^{-1}$ functions. As a consequence, if the constitutive equation is smooth, the velocities need only be $C^{0}$.

Equation (4.3.4) could also be used as the weak form. However, since the derivatives of the stresses would appear in this alternate weak form, the displacements and velocities would have to be $C^{1}$ functions (see Chapter 2); $C^{1}$ functions are difficult to construct in more than one dimension. Furthermore, the trial functions would then have to be constructed so as to satisfy the traction boundary conditions, which would be very difficult. The removal of the derivative of the stresses through integration by parts also leads to certain symmetries in the linearized equations, as will be seen in Chapter 6. Thus the integration by parts is a key step in the development of the weak form.

Next we started with the weak form and showed that it implies the strong form. This, combined with the development of the weak form from the strong form, shows that the weak and strong forms are equivalent. Therefore, if the space of test functions is infinite dimensional, a solution to the weak form is a solution of the strong form. However, the test functions used in computational procedures must be finite dimensional. Therefore, satisfying the weak form in a computation only leads to an approximate solution of the strong form. In linear finite element analysis, it has been shown that the solution of the weak form is the best solution in the sense that it minimizes the error in energy, Strang and Fix (1973). In nonlinear problems, such optimality results are not available in general.
4.3.3. Physical Names of Virtual Power Terms. We will next ascribe a physical name to each of the terms in the virtual power equation. This will be useful in systematizing the development of finite element equations. The nodal forces in the finite element discretization will be identified according to the same physical names.

To identify the first integrand in (4.3.9), note that it can be written as

$$
\begin{equation*}
\frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{j i}=\delta L_{i j} \sigma_{j i}=\left(\delta D_{i j}+\delta W_{i j}\right) \sigma_{j i}=\delta D_{i j} \sigma_{j i}=\delta \mathbf{D}: \sigma \tag{4.3.18}
\end{equation*}
$$

Here we have used the decomposition of the velocity gradient into its symmetric and skew symmetric parts and that $\delta W_{i j} \sigma_{i j}=0$ since $\delta W_{i j}$ is skew symmetric while $\sigma_{i j}$ is symmetric. Comparison with (B4.1.4) then indicates that we can interpret $\delta D_{i j} \sigma_{i j}$ as the rate of virtual internal work, or the virtual internal power, per unit volume. Observe that $\dot{w}^{\text {int }}$ in ( B 4.1 .4 ) is power per unit mass, so $\rho \dot{w}^{\text {int }}=\mathbf{D}: \sigma$ is the power per unit volume. The total virtual internal power $\delta P^{i n t}$ is defined by the integral of $\delta D_{i j} \sigma_{i j}$ over the domain, i.e.

$$
\begin{equation*}
\delta \mathcal{P}^{i n t}=\int_{\Omega} \delta D_{i j} \sigma_{i j} d \Omega=\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{i j} d \Omega \equiv \int_{\Omega} \delta L_{i j} \sigma_{i j} d \Omega=\int_{\Omega} \delta \mathbf{D}: \sigma d \Omega \tag{4.3.19}
\end{equation*}
$$

where the third and fourth terms have been added to remind us that they are equivalent to the second term because of the symmetry of the Cauchy stress tensor.

The second and third terms in (4.3.9) are the virtual external power:

$$
\begin{equation*}
\delta \mathcal{P}^{e x t}=\int_{\Omega} \delta v_{i} \rho b_{i} d \Omega+\sum_{j=1}^{n_{S D}} \int_{\Gamma_{i j}} \delta v_{j} \bar{t}_{j} d \Gamma=\int_{\Omega} \delta \mathbf{v} \cdot \rho \mathbf{b} d \Omega+\sum_{j=1}^{n_{S D}} \int_{\Gamma_{t_{j}}} \delta v_{j} \mathbf{e}_{j} \cdot \overline{\mathbf{t}} d \Gamma \tag{4.3.20}
\end{equation*}
$$

This name is selected because the virtual external power arises from the external body forces $\mathbf{b}(\mathbf{x}, t)$ and prescribed tractions $\mathfrak{t}(x, t)$.

The last term in (4.3.9) is the virtual inertial power

$$
\begin{equation*}
\delta \mathscr{P}^{\text {inert }}=\int_{\Omega} \delta v_{i} \rho \dot{v}_{i} d \Omega \tag{4.3.21}
\end{equation*}
$$

which is the power corresponding to the inertial force. The inertial force can be considered a body force in the d'Alembert sense.

Inserting Eqs. (4.3.19-4.3.21) into (4.3.9), we can write the principle of virtual power as

$$
\begin{equation*}
\delta \mathcal{P}=\delta P^{i n t}-\delta P^{e x t}+\delta \mathcal{P}^{\text {inert }}=0 \quad \forall \delta v_{i} \in \mathcal{U}_{0} \tag{4.3.22}
\end{equation*}
$$

which is the weak form for the momentum equation. The physical meanings help in remembering the weak form and in the derivation of the finite element equations. The weak form is summarized in Box 4.2.

## BOX 4.2

## Weak Form in Updated Lagrangian Formulation: <br> Principle of Virtual Power

If $\sigma_{i j}$ is a smooth function of the displacements and velocities and $v_{i} \in \mathcal{U}$, then if

$$
\begin{equation*}
\delta P^{i n t}-\delta P^{e x t}+\delta \mathcal{P}^{i n e r t}=0 \quad \forall \delta v_{i} \in \mathcal{U}_{0} \tag{B4.2.1}
\end{equation*}
$$

then

$$
\begin{align*}
& \frac{\partial \sigma_{j i}}{\partial x_{j}}+\rho b_{i}=\rho \dot{v}_{i} \text { in } \Omega  \tag{B4.2.2}\\
& n_{j} \sigma_{j i}=\bar{t}_{i} \text { on } \Gamma_{t i}  \tag{B4.2.3}\\
& \left\langle n_{j} \sigma_{j i}\right\rangle=0 \text { on } \Gamma_{i n t} \tag{B4.2.4}
\end{align*}
$$

where

$$
\begin{equation*}
\delta P^{i n t}=\int_{\Omega} \delta \mathbf{D}: \delta d \Omega=\int_{\Omega} \delta D_{i j} \sigma_{i j} d \Omega=\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{i j} d \Omega \tag{B4.2.5}
\end{equation*}
$$

$$
\begin{align*}
& \delta P^{e x t}=\int_{\Omega} \delta \mathbf{v} \cdot \rho \mathbf{b} d \Omega+\sum_{j=1}^{n_{S D}} \int_{\Gamma_{t_{j}}}\left(\delta \mathbf{v} \cdot \mathbf{e}_{j}\right) \overline{\mathbf{t}} \cdot \mathbf{e}_{j} d \Gamma=\int_{\Omega} \delta v_{i} \rho b_{i} d \Omega+\sum_{j=1}^{n_{S D}} \int_{\Gamma_{t_{j}}} \delta v_{j} \bar{t}_{j} d \Gamma  \tag{B4.2.6}\\
& \delta P^{\text {inert }}=\int_{\Omega} \delta \mathbf{v} \cdot \rho \dot{\mathbf{v}} d \Omega=\int_{\Omega} \delta v_{i} \rho \dot{\nu} \dot{i}_{i} d \Omega \tag{B4.2.7}
\end{align*}
$$

### 4.4 UPDATED LAGRANGIAN FINITE ELEMENT DISCRETIZATION

4.4.1 Finite Element Approximation. In this section, the finite element equations for the updated Lagrangian formulation are developed by means of the principle of virtual power. For this purpose the current domain $\Omega$ is subdivided into elements $\Omega_{e}$ so that the union of the elements comprises the total domain, $\Omega=\bigcup_{e} \Omega_{e}$. The nodal coordinates in the current configuration are denoted by $x_{i I}, I=1$ to $n_{N}$. Lower case subscripts are used for components, upper case subscripts for nodal values. In two dimensions, $x_{i I}=\left[x_{I}, y_{I}\right]$, in three dimensions $x_{i I}=\left[x_{I}, y_{I}, z_{I}\right]$. The nodal coordinates in the undeformed configuration are $X_{i I}$.

In the finite element method, the motion $\mathbf{x}(\mathbf{X}, t)$ is approximated by

$$
\begin{equation*}
x_{i}(\mathbf{X}, t)=N_{I}(\mathbf{X}) x_{i I}(t) \quad \text { or } \quad \mathbf{x}(\mathbf{X}, t)=N_{I}(\mathbf{X}) \mathbf{x}_{I}(t) \tag{4.4.1}
\end{equation*}
$$

where $N_{I}(\mathbf{X})$ are the interpolation (shape) functions and $\mathbf{x}_{I}$ is the position vector of node $I$. Summation over repeated indices is implied; in the case of lower case indices, the sum is over the number of space dimensions, while for upper case indices the sum is over the number of nodes. The nodes in the sum depends on the entity considered: when the total domain is considered, the sum is over all nodes in the domain, whereas when an element is considered, the sum is over the nodes of the element.

Writing (4.4.1) at a node with initial position $\mathbf{X}_{J}$ we have

$$
\begin{equation*}
\mathbf{x}\left(\mathbf{X}_{J}, t\right)=\mathbf{x}_{I}(t) N_{I}\left(\mathbf{X}_{J}\right)=\mathbf{x}_{I}(t) \delta_{I J}=\mathbf{x}_{J}(t) \tag{4.4.3}
\end{equation*}
$$

where we have used the interpolation property of the shape functions in the third term. Interpreting this equation, we see that node $J$ always corresponds to the same material point $\mathbf{X}_{J}$ : in a Lagrangian mesh, nodes remain coincident with material points.

We define the nodal displacements by using Eq. (3.2.7) at the nodes

$$
\begin{equation*}
u_{i I}(t)=x_{i I}(t)-X_{i I} \quad \text { or } \quad \mathbf{u}_{I}(t)=\mathbf{x}_{I}(t)-\mathbf{X}_{I} \tag{4.4.4a}
\end{equation*}
$$

The displacement field is

$$
\begin{equation*}
u_{i}(\mathbf{X}, t)=x_{i}(\mathbf{X}, t)-X_{i}=u_{i I}(t) N_{I}(\mathbf{X}) \quad \text { or } \quad \mathbf{u}(\mathbf{X}, t)=\mathbf{u}_{I}(t) N_{I}(\mathbf{X}) \tag{4.4.4b}
\end{equation*}
$$

which follows from (4.4.1), (4.4.2) and (4.4.3).
The velocities are obtained by taking the material time derivative of the displacements, giving

$$
\begin{equation*}
v_{i}(\mathbf{X}, t)=\frac{\partial u_{i}(\mathbf{X}, t)}{\partial t}=\dot{u}_{i I}(t) N_{I}(\mathbf{X})=v_{i I}(t) N_{I}(\mathbf{X}) \quad \text { or } \quad \mathbf{v}(\mathbf{X}, t)=\dot{\mathbf{u}}_{I}(t) N_{I}(\mathbf{X}) \tag{4.4.5}
\end{equation*}
$$

where we have written out the derivative of the displacement on the left hand side to stress that the velocity is a material time derivative of the displacement, i.e., the partial derivative with respect to time with the material coordinate fixed. Note the velocities are given by the same shape function since the shape functions are constant in time. The superposed dot on the nodal displacements is an ordinary derivative, since the nodal displacements are only functions of time.

The accelerations are similarly given by the material time derivative of the velocities

$$
\begin{equation*}
\ddot{u}_{i}(\mathbf{X}, t)=\ddot{u}_{i I}(t) N_{I}(\mathbf{X}) \quad \text { or } \quad \ddot{\mathbf{u}}(\mathbf{X}, t)=\ddot{\mathbf{u}}_{I}(t) N_{I}(\mathbf{X}) \tag{4.4.6}
\end{equation*}
$$

It is emphasized that the shape functions are expressed in terms of the material coordinates in the updated Lagrangian formulation even though we will use the weak form in the current configuration. As pointed out in Section 2.8, it is crucial to express the shape functions in terms of material coordinates when a Lagrangian mesh is used because we want the time dependence in the finite element approximation of the motion to reside entirely in the nodal variables.

The velocity gradient is obtained by substituting Eq. (4.4.5) into Eq. (3.3.7), which yields

$$
\begin{equation*}
L_{i j}=v_{i, j}=v_{i I} \frac{\partial N_{I}}{\partial x_{j}}=v_{i I} N_{I, j} \quad \text { or } \quad \mathbf{L}=\mathbf{v}_{I} N_{I, j} \tag{4.4.7}
\end{equation*}
$$

and the rate-of-deformation is given by

$$
\begin{equation*}
D_{i j}=\frac{1}{2}\left(L_{i j}+L_{j i}\right)=\frac{1}{2}\left(v_{i I} N_{I, j}+v_{j I} N_{I, i}\right) \tag{4.4.7b}
\end{equation*}
$$

In the construction of the finite element approximation to the motion, Eq. (4.4.1), we have ignored the velocity boundary conditions, i.e. the velocities given by Eq. (4.4.5) are not in the space defined by Eq. (4.3.2). We will first develop the equations for an unconstrained body with no velocity boundary conditions, and then modify the discrete equations to account for the velocity boundary conditions.

In Eq. (4.4.1), all components of the motion are approximated by the same shape functions. This construction of the motion facilitates the representation of rigid body rotation, which is an essential requirement for convergence. This is discussed further in Chapter 8.

The test function, or variation, is not a function of time, so we approximate the test function as

$$
\begin{equation*}
\delta v_{i}(\mathbf{X})=\delta v_{i I} N_{I}(\mathbf{X}) \quad \text { or } \quad \delta \mathbf{v}(\mathbf{X})=\delta \mathbf{v}_{I} N_{I}(\mathbf{X}) \tag{4.4.8}
\end{equation*}
$$

where $\delta v_{i I}$ are the virtual nodal velocities.

As a first step in the construction of the discrete finite element equations, the test function is substituted into principle of virtual power giving

$$
\begin{equation*}
\delta v_{i I} \int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega-\delta v_{i I} \int_{\Omega} N_{I} \rho b_{i} d \Omega-\sum_{i=1}^{n_{S D}} \delta v_{i I} \int_{\Gamma_{t_{i}}} N_{I} \bar{t}_{i} d \Gamma+\delta v_{i I} \int_{\Omega} N_{I} \rho \dot{v}_{i} d \Omega=0 \tag{4.4.9a}
\end{equation*}
$$

The stresses in (4.4.9a) are functions of the trial velocities and trial displacements. From the definition of the test space, (4.3.4), the virtual velocities must vanish wherever the velocities are prescribed, i.e. $\delta v_{i}=0$ on $\Gamma_{v_{i}}$ and therefore only the virtual nodal velocities for nodes not on $\Gamma_{v_{i}}$ are arbitrary, as indicated above. Using the arbitrariness of the virtual nodal velocities everywhere except on $\Gamma_{v_{i}}$, it then follows that the weak form of the momentum equation is

$$
\begin{equation*}
\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega-\int_{\Omega} N_{I} \rho b_{i} d \Omega-\sum_{j=1}^{n_{S D}} \int_{\Gamma_{\Gamma_{j}}} N_{I} \bar{t}_{i} d \Gamma+\int_{\Omega} N_{I} \rho \dot{v_{i}} d \Omega=0 \quad \forall I, i \notin \Gamma_{v_{i}} \tag{4.4.9b}
\end{equation*}
$$

However, the above form is difficult to remember. For purposes of convenience and for a better physical interpretation, it is worthwhile to ascribe physical names to each of the terms in the above equation.
4.4.2. Internal and External Nodal Forces. We define the nodal forces corresponding to each term in the virtual power equation. This helps in remembering the equation and also provides a systematic procedure which is found in most finite element software. The internal nodal forces are defined by

$$
\begin{equation*}
\delta P^{i n t}=\delta v_{i l} f_{i I}^{i n t}=\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{j i} d \Omega=\delta v_{i l} \int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega \tag{4.4.10}
\end{equation*}
$$

where the third term is the definition of internal virtual power as given in Eqs. (B4.2.5) and (4.4.8) has been used in the last term. From the above it can be seen that the internal nodal forces are given by

$$
\begin{equation*}
f_{i I}^{\text {int }}=\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega \tag{4.4.11}
\end{equation*}
$$

These nodal forces are called internal because they represent the stresses in the body. These expressions apply to both a complete mesh and to any element or group of elements, as has been described in Chapter 2. Note that this expression involves derivatives of the shape functions with respect to spatial coordinates and integration over the current configuration. Equation (4.4.11) is a key equation in nonlinear finite element methods for updated Lagrangian meshes; it applies also to Eulerian and ALE meshes.

The external nodal forces are defined similarly in terms of the virtual external power

$$
\begin{align*}
& \delta \mathcal{P}^{e x t}=\delta v_{i I} f_{i I}^{e x t}=\int_{\Omega} \delta v_{i} \rho b_{i} d \Omega+\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t_{i}}} \delta v_{i} \bar{t}_{i} d \Gamma  \tag{4.4.12}\\
& =\delta v_{i I} \int_{\Omega} N_{I} \rho b_{i} d \Omega+\sum_{i=1}^{n_{S D}} \delta v_{i I} \int_{\Gamma_{t_{i}}} N_{I} \bar{t}_{i} d \Gamma
\end{align*}
$$

so the external nodal forces are given by

$$
\begin{equation*}
f_{i l}^{e x t}=\int_{\Omega} N_{I} \rho b_{i} d \Omega+\int_{\Gamma_{i_{i}}} N_{I} \bar{t} d \Gamma \quad \text { or } \quad \mathbf{f}_{I}^{e x t}=\int_{\Omega} N_{I} \rho \mathbf{b} d \Omega+\int_{\Gamma_{t_{i}}} N_{I} \mathbf{e}_{i} \cdot \overline{\mathbf{t}} d \Gamma \tag{4.4.13}
\end{equation*}
$$

4.4.3. Mass Matrix and Inertial Forces. The inertial nodal forces are defined by

$$
\begin{equation*}
\delta P^{\text {inert }}=\delta v_{i I} f_{i I}^{\text {inert }}=\int_{\Omega} \delta v_{i} \rho \dot{v}_{i} d \Omega=\delta v_{i I} \int_{\Omega} N_{I} \rho \dot{v}_{i} d \Omega \tag{4.4.14}
\end{equation*}
$$

so

$$
\begin{equation*}
f_{i I}^{\text {inert }}=\int_{\Omega} \rho N_{I} \dot{v}_{i} d \Omega \quad \text { or } \quad \mathbf{f}_{I}^{\text {inert }}=\int_{\Omega} \rho N_{I} \dot{\mathbf{v}} d \Omega \tag{4.4.15}
\end{equation*}
$$

Using the expression (4.4.6) for the accelerations in the above gives

$$
\begin{equation*}
f_{i I}^{\text {inert }}=\int_{\Omega} \rho N_{I} N_{J} d \Omega \dot{v}_{i J} \tag{4.4.16}
\end{equation*}
$$

It is convenient to define these nodal forces as a product of a mass matrix and the nodal accelerations. Defining the mass matrix by

$$
\begin{equation*}
M_{i j J}=\delta_{i j} \int_{\Omega} \rho N_{I} N_{J} d \Omega \tag{4.4.17}
\end{equation*}
$$

it follows from (4.4.16) and (4.4.17) that the inertial forces are given by

$$
\begin{equation*}
f_{i I}^{\text {inert }}=M_{i j I J} \dot{v}_{j J} \quad \text { or } \quad \mathbf{f}_{I}^{\text {inert }}=\mathbf{M}_{I J} \dot{\mathbf{v}}_{J} \tag{4.4.18}
\end{equation*}
$$

4.4.4. Discrete Equations. With the definitions of the internal, external and inertial nodal forces, Eqs. (4.4.10), (4.4.12) and (4.4.17), we can concisely write the discrete approximation to the weak form (4.4.9a) as

$$
\begin{equation*}
\delta v_{i l}\left(f_{i I}^{i n t}-f_{i I}^{e x t}+M_{i j J J} \dot{v}_{j J}\right)=0 \quad \text { for } \quad \forall \delta v_{i I} \notin \Gamma_{v_{i}} \tag{4.4.19}
\end{equation*}
$$

Invoking the arbitrariness of the unconstrained, virtual nodal velocities gives

$$
\begin{equation*}
M_{i j J J} \dot{v}_{j J}+f_{i I}^{\text {int }}=f_{i I}^{\text {ext }} \quad \forall I, i \notin \Gamma_{v_{i}} \quad \text { or } \quad \mathbf{M}_{I J} \dot{\mathbf{v}}_{J}+\mathbf{f}_{I}^{\text {int }}=\mathbf{f}_{I}^{e x t} \tag{4.4.20}
\end{equation*}
$$

The above are the discrete momentum equations or the equations of motion; they are also called the semidiscrete momentum equations since they have not been discretized in time. The implicit sums are over all components and all nodes of the mesh; any prescribed velocity component that appears in the above is not an unknown. The matrix form on the left depends on the interpretation of the indices: this is discussed further in Section 4.5.

The semidiscrete momentum equations are a system of $n_{D O F}$ ordinary differential equations in the nodal velocities, where $n_{D O F}$ is the number of nodal velocity components which are unconstrained; $n_{D O F}$ is often called the number of degrees of freedom. To complete the system of equations, we append the constitutive equations at the element quadrature points and the expression for the rate-of-deformation in terms of the nodal velocities. Let the $n_{Q}$ quadrature points in the mesh be denoted by

$$
\begin{equation*}
\mathbf{x}_{Q}(t)=N_{I}\left(\mathbf{X}_{Q}\right) \mathbf{x}_{I}(t) \tag{4.4.21}
\end{equation*}
$$

Note that the quadrature points are coincident with material points. Let $n_{\sigma}$ be the number of independent components of the stress tensor: in a two dimensional plane stress problem, $n_{\sigma}=3$, since the stress tensor $\sigma$ is symmetric; in three-dimensional problems, $n_{\sigma}=6$.

The semidiscrete equations for the finite element approximation then consist of the following ordinary differential equations in time:

$$
\begin{align*}
& M_{i j J} \dot{v}_{j J}+f_{i I}^{i n t}=f_{i I}^{e x t} \quad \text { for } \quad(I, i) \notin \Gamma_{v_{i}}  \tag{4.4.22}\\
& \sigma_{i j}^{\nabla}\left(\mathbf{X}_{Q}\right)=S_{i j}\left(D_{k l}\left(\mathbf{x}_{Q}\right), \text { etc }\right) \quad \forall \mathbf{X}_{Q}  \tag{4.4.23}\\
& \text { where } D_{i j}\left(\mathbf{X}_{Q}\right)=\frac{1}{2}\left(L_{i j}+L_{j i}\right) \quad \text { and } \quad L_{i j}=N_{I, j}\left(\mathbf{X}_{Q}\right) v_{i I} \tag{4.4.24}
\end{align*}
$$

This is a standard initial value problem, consisting of first-order ordinary differential equations in the velocities $v_{i I}(t)$ and the stresses $\sigma_{i j}\left(\mathbf{X}_{Q}, t\right)$. If we substitute (4.4.24) into (4.4.23) to eliminate the rate-of-deformation from the equations, the total number of unknowns is $n_{D O F}+n_{\sigma} n_{Q}$. This system of ordinary differential equations can be integrated in time by any of the methods for integrating ordinary differential equations, such as Runge-Kutta methods or the central difference method; this is discussed in Chapter 6.

The nodal velocities on prescribed velocity boundaries, $v_{i I},(I, i) \in \Gamma_{v_{i}}$, are obtained from the boundary conditions, Eq. (B4.1.7b). The initial conditions (B4.1.9) are applied at the nodes and quadrature points

$$
\begin{align*}
& v_{i I}(0)=v_{i I}^{0}  \tag{4.4.25}\\
& \sigma_{i j}\left(\mathbf{x}_{Q}, 0\right)=\sigma_{i j}^{0}\left(\mathbf{x}_{Q}\right) \tag{4.4.26}
\end{align*}
$$

where $v_{i I}^{0}$ and $\sigma_{i j}^{0}$ are initial data at the nodes and quadrature points. If data for the initial conditions are given at a different set of points, the values at the nodes and quadrature points can be estimated by least square fits, as in Section 2.4.5.

For an equilibrium problem, the accelerations vanish and the governing equations are

$$
\begin{equation*}
f_{i I}^{i n t}=f_{i I}^{e x t} \quad \text { for } \quad(I, i) \notin \Gamma_{v_{i}} \quad \text { or } \quad \mathbf{f}^{i n t}=\mathbf{f}^{e x t} \tag{4.4.27}
\end{equation*}
$$

along with (4.4.23) and (4.4.24). The above are called the discrete equilibrium equations. If the constitutive equations are rate-independent, then the discrete equilibrium equations are a set of nonlinear algebraic equations in the stresses and nodal displacements. For rate-dependent materials, any rate terms must be discretized in time to obtain a set of nonlinear algebraicequations; this is further discussed in Chapter 6.
4.4.5. Element Coordinates. Finite elements are usually developed with shape functions expressed in terms of parent element coordinates, which we will often call element coordinates for brevity. Examples of element coordinates are triangular coordinates and isoparametric coordinates. We will next describe the use of shape functions expressed in terms of element coordinates. As part of this description, we will show that the element coordinates can be considered an alternative set of material coordinates in a Lagrangian mesh. Therefore, expressing the shape functions in terms of element coordinates is intrinsically equivalent to expressing them in terms of material coordinates. We denote the parent element coordinates by $\xi_{i}^{e}$, or $\xi^{e}$ in tensor notation, and the parent domain by $\Delta$; the superscript $e$ will only be carried in the beginning of this description. The shape of the parent domain depends on the type of element and the dimension of the problem; it may be a biunit square, a triangle, or a cube, for example. Specific parent domains are given in the examples which follow.

When a Lagrangian element is treated in terms of element coordinates, we are concerned with three domains that correspond to an element:

1. the parent element domain $\Delta$;
2. the current element domain $\Omega^{e}=\Omega^{e}(t)$;
3. the initial (reference) element domain $\Omega_{0}^{e}$

The following maps are pertinent:

1. parent domain to current configuration: $\mathbf{x}=\mathbf{x}\left(\xi^{e}, t\right)$
2. parent domain to initial configuration: $\mathbf{X}=\mathbf{X}\left(\xi^{e}\right)$
3. initial configuration to the current configuration, i.e. the motion $\mathbf{x}=\mathbf{x}(\mathbf{X}, t) \equiv \phi(\mathbf{X}, t)$

The map $\mathbf{X}=\mathbf{X}\left(\xi^{e}\right)$ corresponds to $\mathbf{x}=\mathbf{x}\left(\xi^{e}, 0\right)$. These maps are illustrated in Fig. 4.1 for a triangular element where a space-time plot of a two-dimensional triangular element is shown.

The motion in each element is described by a composition of these maps

$$
\begin{equation*}
\mathbf{x}=\mathbf{x}(\mathbf{X}, t)=\mathbf{x}\left(\xi^{e}(\mathbf{X}), t\right) \quad \mathbf{x}(\mathbf{X}, t)=\mathbf{x}\left(\xi^{e}, t\right) \circ \xi^{e}(\mathbf{X}) \text { in } \Omega_{e} \tag{4.4.28}
\end{equation*}
$$

where $\xi^{e}(\mathbf{X})=\mathbf{X}^{-1}\left(\xi^{e}\right)$. For the motion to be well defined and smooth, the inverse map $\mathbf{X}^{-1}\left(\xi^{e}\right)$ must exist and the function $\mathbf{x}=\mathbf{x}\left(\xi^{e}, t\right)$ must be sufficiently smooth and meet certain conditions of regularity so that $\mathbf{x}^{-1}\left(\xi^{e}, t\right)$ exists; these conditions are given in Section 4.4.8. The inverse map $\mathbf{x}^{-1}\left(\xi^{e}, t\right)$ is usually not constructed because in most cases it cannot be obtained explicitly, so instead the derivatives with respect to the spatial coordinates are obtained in terms of the derivatives with respect to the parent coordinates by implicit differentiation.

The motion is approximated by

$$
\begin{equation*}
x_{i}(\xi, t)=x_{i I}(t) N_{I}(\xi) \quad \text { or } \quad \mathbf{x}(\xi, t)=\mathbf{x}_{I}(t) N_{I}(\xi) \tag{4.4.29}
\end{equation*}
$$

where we have dropped the supercript $e$ on the element coordinates. As can be seen in the above, the shape functions $N_{I}(\xi)$ are only functions of the parent element coordinates; the time dependence of the motion resides entirely in the nodal coordinates. The above represents a time dependent mapping between the parent domain and the current configuration of the element.

Writing this map at time $t=0$ we obtain

$$
\begin{equation*}
X_{i}(\xi)=x_{i}(\xi, 0)=x_{i I}(0) N_{I}(\xi)=X_{i I} N_{I}(\xi) \quad \text { or } \quad \mathbf{X}(\xi)=\mathbf{X}_{I} N_{I}(\xi) \tag{4.4.30}
\end{equation*}
$$

It can be seen from (4.4.30) that the map between the material coordinates and the element coordinates is time invariant in a Lagrangian element. If this map is one-to-one and onto, then the element coordinates can in fact be considered surrogate material coordinates in a Lagrangian mesh, since each material point in an element then has a unique element coordinate label. To establish a unique correspondence between element coordinates and the material coordinates in $\Omega_{0}$, the element number must be part of the label. This does not apply to meshes which are not Lagrangian, as will be seen in Chapter 7. The use of the initial coordinates $\mathbf{X}$ as material coordinates in fact originates mainly in analysis; in finite element methods, the use of element coordinates as material labels is more natural.

As before, since the element coordinates are time invariant, we can express the displacements, velocities and accelerations in terms of the same shape functions

$$
\begin{array}{ll}
u_{i}(\xi, t)=u_{i I}(t) N_{I}(\xi) & \mathbf{u}(\xi, t)=\mathbf{u}_{I}(t) N_{I}(\xi) \\
\dot{u}_{i}(\xi, t)=v_{i}(\xi, t)=v_{i I}(t) N_{I}(\xi) & \mathbf{u}(\xi, t)=\mathbf{v}(\xi, t)=\mathbf{v}_{I}(t) N_{I}(\xi) \\
\dot{v}(\xi, t)=\dot{v}_{i I}(t) N_{I}(\xi) & \dot{\mathbf{v}}(\xi, t)=\dot{\mathbf{v}}_{I}(t) N_{I}(\xi)
\end{array}
$$

where we have obtained (4.4.32) by taking material time derivative of (4.4.31) and we have obtained (4.4.33) by taking the material time derivative of (4.4.32). The time dependence, as before, resides entirely in the nodal variables, since the element coordinates are independent of time.


Fig. 4.1. Initial and current configurations of an element and their relationships to the parent element.
4.4.6. Derivatives of Functions. The spatial derivatives of the velocity field are obtained by implicit differentiation because the function $\mathbf{x}(\xi, t)$ is generally not explicitly invertible; i.e. it is not possible to write closed-form expressions for $\xi$ in terms of $\mathbf{x}$. By the chain rule,

$$
\begin{equation*}
\frac{\partial v_{i}}{\partial \xi_{j}}=\frac{\partial v_{i}}{\partial x_{k}} \frac{\partial x_{k}}{\partial \xi_{j}} \quad \text { or } \quad \mathbf{v}_{, \xi}=\mathbf{v}_{\mathbf{x}} \mathbf{x}_{, \xi} \tag{4.4.34}
\end{equation*}
$$

The matrix $\partial x_{k} / \partial \xi_{j}$ is the Jacobian of the map between the current configuration of the element and the parent element configuration. We will use two symbols for this matrix: $\mathbf{x}_{\xi}$ and $\mathbf{F}_{\xi}$, where $F_{i j}^{\xi}=\partial x_{i} / \partial \xi_{j}$. The second symbol is used to convey the notion that the Jacobian with respect to the element coordinates can be viewed as a deformation gradient with respect to the parent element configuration. In two dimensions

$$
\mathbf{x}_{\xi}(\xi, t) \equiv \mathbf{F}_{\xi}(\xi, t)=\left[\begin{array}{ll}
x_{\xi_{1}} & x_{\xi_{2}}  \tag{4.4.35}\\
y_{, \xi_{1}} & y_{\xi_{2}}
\end{array}\right]
$$

As indicated in (4.4.35), the Jacobian of the map between the current and parent configurations is a function of time.

Inverting (4.4.34), we obtain

$$
\begin{equation*}
L_{i j}=\frac{\partial v_{i}}{\partial \xi_{k}} F_{\xi k j}^{-1}=\left(\frac{\partial v_{i}}{\partial \xi_{k}}\right)\left(\frac{\partial \xi_{k}}{\partial x_{j}}\right) \quad \text { or } \quad \mathbf{L}=\mathbf{v}, \mathbf{x}=\mathbf{v}, \xi \mathbf{x}, \xi_{\xi}^{-1}=\mathbf{v}, \xi \mathbf{F}_{\xi}^{-1} \tag{4.4.36}
\end{equation*}
$$

Thus computation the derivatives with respect to $\xi$ involves finding the inverse of the Jacobian between the current and parent element coordinates; the matrix to be inverted in the two dimensional case is given in (4.4.35). Similarly for the shape functions $N_{I}$, we have

$$
\begin{equation*}
N_{I, \mathbf{x}}^{T}=N_{I, \xi}^{T} \mathbf{x}_{, \xi}^{-1}=N_{I, \xi}^{T} \mathbf{F}_{\xi}^{-1} \tag{4.4.37}
\end{equation*}
$$

where the transpose appears in the matrix expressions because we consider $N_{I, \mathbf{x}}$ and $N_{I, \xi}$ to be column matrices and the matrix on the RHS of the above must be a row matrix. The determinant of the element Jacobian $\mathbf{F}_{\xi}$,

$$
\begin{equation*}
J_{\xi}=\operatorname{det}(\mathbf{x}, \xi) \tag{4.4.38}
\end{equation*}
$$

is called the element Jacobian determinant; we append the subscript to distinguish it from the determinant of the deformation gradient, $J$. Substituting (4.4.37) into (4.4.36) gives

$$
\begin{equation*}
L_{i j}=v_{i I} \frac{\partial N_{I}}{\partial \xi_{k}} F_{\xi k j}^{-1} \quad \text { or } \quad \mathbf{L}=\mathbf{v}_{I} N_{I, \xi}^{T} \mathbf{x}_{\xi}^{-l} \tag{4.4.39}
\end{equation*}
$$

The rate-of-deformation is obtained from the velocity gradient by using (3.3.10).
4.4.7. Integration and Nodal Forces. Integrals on the current configuration are related to integrals over the reference domain and the parent domain by

$$
\begin{equation*}
\int_{\Omega^{e}} g(\mathbf{x}) d \Omega=\int_{\Omega_{0}^{e}} g(\mathbf{X}) J d \Omega_{0}=\int_{\Delta} g(\xi) J_{\xi} d \Delta \quad \text { and } \quad \int_{\Omega_{0}^{e}} g(\mathbf{X}) d \Omega_{0}=\int_{\Delta} g(\xi) J_{\xi}^{0} d \Delta \tag{4.4.40}
\end{equation*}
$$

where $J$ and $J_{\xi}$ are the determinants of the Jacobians between the current configurations and the reference and parent element configurations, respectively. Part of equations is identical to (3.2.18). The other part is obtained in the same way by using the map between the current and parent configurations. The above is consitent with our convention

When the internal nodal forces are computed by integration over the parent domain, (4.4.11) is tranformed to the parent element domain by (4.4.40), giving

$$
\begin{equation*}
f_{i l}^{i n t}=\int_{\Omega^{e}} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega=\int_{\Delta} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} J_{\xi} d \Delta \tag{4.4.41}
\end{equation*}
$$

The external nodal forces and the mass matrix can similarly be integrated over the parent domain.
4.4.8. Conditions on Parent to Current Map. The finite element approximation to the motion $\mathbf{x}(\xi, t)$, which maps the parent domain of an element onto the current domain of the
element, is subject to the same conditions as $\phi(\mathbf{X}, t)$, as given in Section 3.3.6, except that no discontinuities are allowed. These conditions are:

1. $\mathbf{x}(\xi, t)$ must be one-to-one and onto;
2. $\mathbf{x}(\xi, t)$ must be at least $C^{0}$ in space;
3. the element Jacobian determinant must be positive, i.e.

$$
\begin{equation*}
J_{\xi} \equiv \operatorname{det}\left(\mathbf{x}_{, \xi}\right)>0 . \tag{4.4.43}
\end{equation*}
$$

These conditions insure that $\mathbf{x}(\xi, t)$ is invertible.
We now explain why the condition $\operatorname{det}(\mathbf{x}, \xi)>0$ is necessary. We first use the chain rule to express $\mathbf{x}_{\xi}$ in terms of $\mathbf{F}$ and $\mathbf{X}_{\xi}$ :

$$
\begin{equation*}
\frac{\partial x_{i}}{\partial \xi_{j}}=\frac{\partial x_{i}}{\partial X_{k}} \frac{\partial X_{k}}{\partial \xi_{j}}=F_{i k} \frac{\partial X_{k}}{\partial \xi_{j}} \quad \text { or } \quad \mathbf{x}_{\xi \xi}=\mathbf{x}_{/ \mathbf{X}} \mathbf{X}_{, \xi}=\mathbf{F} \mathbf{X}_{\xi \xi} \tag{4.4.44a}
\end{equation*}
$$

We can also write the above as

$$
\begin{equation*}
\mathbf{F}_{\xi}=\mathbf{F} \cdot \mathbf{F}_{\xi}^{0} \tag{4.4.44b}
\end{equation*}
$$

which highlights the fact that the deformation gradient with respect to the parent element coordinates is the product of the standard deformation gradient and the initial deformation gradient with respect to the parent element coordinates. The determinant of the product of two matrices is equal to the product of the determinants, so

$$
\begin{equation*}
\operatorname{det}\left(\mathbf{x}_{, \xi}\right)=\operatorname{det}(\mathbf{F}) \operatorname{det}\left(\mathbf{X}_{, \xi}\right) \equiv J J_{\xi}^{0} \tag{4.4.45}
\end{equation*}
$$

We assume that the elements in the initial mesh are properly constructed so that $J_{\xi}^{0}=J_{\xi}(0)>0$ for all elements; otherwise the initial mapping would not be one-to-one. If $J_{\xi}(t) \leq 0$ at any time then by (4.4.45), $J \leq 0$. By the conservation of matter $\rho=\rho_{0} / J$ so $J \leq 0$ implies $\rho \leq 0$, which is physically impossible. Therefore it is necessary that $J_{\xi}(t)>0$ for all time. In some calculations, excessive mesh distortion can result in severely deformed meshes in which $J_{\xi} \leq 0$. This implies a negative density, so such calculations violate the physical principle that mass is always positive.
4.4.9. Simplifications of Mass Matrix. When the same shape functions are used for all components, it is convenient to take advantage of the form of the mass matrix (4.4.20) by writing it as

$$
\begin{equation*}
M_{i j J J}=\delta_{i j} \tilde{M}_{I J} \tag{4.4.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{M}_{I J}=\int_{\Omega} \rho N_{I} N_{J} d \Omega \quad \tilde{\mathbf{M}}=\int_{\Omega} \rho \mathbf{N}^{T} \mathbf{N} d \Omega \tag{4.4.47}
\end{equation*}
$$

Then the equations of motion (4.4.22) become

$$
\begin{equation*}
\tilde{M}_{I J} \dot{v}_{i J}+f_{i I}^{i n t}=f_{i I}^{e x t} \tag{4.4.48}
\end{equation*}
$$

This form is advantageous when the consistent mass matrix is used with explicit time integration, since the order of the matrix which needs to be inverted is reduced by a factor of $n_{S D}$.

We next show that the mass matrix for a Lagrangian mesh is constant in time. If the shape functions are expressed in terms of parent element coordinates, then

$$
\begin{equation*}
M_{i j I J}=\delta_{i j} \int_{\Delta} \rho N_{I} N_{J} \operatorname{det}(\mathbf{x}, \xi) d \Delta=\delta_{i j} \int_{\Omega} \rho N_{I} N_{J} d \Omega \tag{4.4.49}
\end{equation*}
$$

Since $\operatorname{det}(\mathbf{x}, \xi)$ and the density are time dependent, this mass matrix appears to be time dependent. To show that the matrix is in fact time independent, we transform the above integral to the undeformed configuration by (3.2.18), giving

$$
\begin{equation*}
M_{i j J J}=\delta_{i j} \int_{\Omega_{0}} \rho N_{I} N_{J} J d \Omega_{0} \tag{4.4.50}
\end{equation*}
$$

From mass conservation, (B4.1.1) it follows that $\rho J=\rho_{0}$. Hence (4.4.50) becomes

$$
\begin{equation*}
M_{i j J J}=\delta_{i j} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0} \quad \text { or } \quad M_{i j J J}=\delta_{i j} \int_{\Delta} \rho_{0} N_{I} N_{J} J_{\xi}^{0} d \Delta \tag{4.4.51}
\end{equation*}
$$

The compact form of the mass matrix, (4.4.47) can similarly be written as

$$
\begin{equation*}
\tilde{M}_{I J}=\int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0} \quad \text { and } \quad \mathbf{M}_{I J}=\mathbf{I} \tilde{M}_{I J}=\mathbf{I} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0} \tag{4.4.52}
\end{equation*}
$$

In the above integrals, the integrand is independent of time, so the mass matrix is constant in time. It needs to be evaluated only at the beginning of a computation. The same result could be obtained by computing the mass matrix by (4.4.49) at the initial time, i.e. in the intial configuration. The mass matrix in (4.4.52) can be called total Lagrangian since it is evaluated in the reference (undeformed) configuration. We take the view here and subsequently that the discrete equations should be evaluated in whatever configuration is most convenient.

### 4.5. IMPLEMENTATION

In the implementation of the finite element equations developed in the previous Section, two approaches are popular:

1. the indicial expressions are directly treated as matrix equations;
2. Voigt notation is used, as in linear finite element methods, so the square stress and strain matrices are converted to column matrices.

## Box 4.3

## Discrete Equations and Internal Nodal Force Algorithm for the Updated Lagrangian Formulation

Equations of Motion (discrete momentum equation)

$$
\begin{equation*}
M_{i j J J} \dot{v}_{j J}+f_{i I}^{\text {int }}=f_{i I}^{\text {ext }} \quad \text { for } \quad(I, i) \notin \Gamma_{v_{i}} \tag{B4.8.1}
\end{equation*}
$$

Internal Nodal Forces

$$
\begin{aligned}
f_{i I}^{i n t} & =\int_{\Omega} \mathcal{B}_{I j} \sigma_{j i} d \Omega=\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega \text { or }\left(\mathbf{f}_{I}^{\text {int }}\right)^{T}=\int_{\Omega} \mathcal{B}_{I}^{T} \sigma d \Omega \\
\mathbf{f}_{I}^{i n t} & =\int_{\Omega} \mathbf{B}_{I}^{T}\{\sigma\} d \Omega \text { in Voigt notation }
\end{aligned}
$$

External Nodal Forces

$$
\begin{equation*}
f_{i l}^{e x t}=\int_{\Omega} N_{I} \rho b_{i} d \Omega+\int_{\Gamma_{L_{i}}} N_{I} \bar{t}_{i} d \Gamma \quad \text { or } \quad \mathbf{f}_{I}^{e x t}=\int_{\Omega} N_{I} \rho \mathbf{b} d \Omega+\int_{\Gamma_{t_{i}}} N_{I} \mathbf{e}_{i} \cdot \overline{\mathbf{t}} d \Gamma \tag{B4.8.3}
\end{equation*}
$$

Mass Matrix (total Lagrangian)

$$
\begin{align*}
& M_{i j I J}=\delta_{i j} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0}=\delta_{i j} \int_{\Delta} \rho_{0} N_{I} N_{J} J_{\xi}^{0} d \Delta  \tag{B4.8.4}\\
& \mathbf{M}_{I J}=\tilde{\mathbf{M}}_{I J}=\mathbf{I} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0} \tag{B4.8.5}
\end{align*}
$$

## Internal nodal force computation for element

1. $\mathbf{f}^{i n t}=0$
2. for all quadrature points $\xi_{Q}$
i. compute $\left[\mathcal{B}_{I j}\right]=\left[\partial N_{I}\left(\xi_{Q}\right) / \partial x_{j}\right]$ for all $I$
ii. $\mathbf{L}=\left[L_{i j}\right]=\left[v_{i I} \mathcal{B}_{I j}\right]=\mathbf{v}_{I} \mathcal{B}_{I}^{T} ; \quad L_{i j}=\frac{\partial N_{I}}{\partial x_{j}} v_{i I}$
iii. $\mathbf{D}=\frac{1}{2}\left(\mathbf{L}^{T}+\mathbf{L}\right)$
iv. if needed compute $\mathbf{F}$ and $\mathbf{E}$ by procedures in Box 4.7
v. compute the Cauchy stress $\sigma$ or the PK2 stress $\mathbf{S}$ or by constitutive equation
vi. if $\mathbf{S}$ computed, compute $\sigma$ by $\sigma=J^{-1} \mathbf{F S} \mathbf{F}^{T}$
vii. $\mathbf{f}_{I}^{i n t} \leftarrow \mathbf{f}_{I}^{\text {int }}+\mathcal{B}_{I}^{T} \sigma J_{\xi} \bar{w}_{Q}$ for all I
end loop
$\bar{w}_{Q}$ are quadrature weights

Each of these methods has advantages, so both methods will be described. In Box 4.8 the discrete eqautions are summarized in both forms. The internal force computations is then given for the matrix-indicial form.
4.5.1. Indicial to Matrix Expressions. The conversion of indicial expressions to matrix frorm is somewhat arbitrary and depends on individual preferences. In this book, we have tried to interpret single-index variables as column matrices in most cases; the details are somewhat different when there is a preference for row matrices. To illustrate this procedure, consider the expression for the velocity gradient, Eq. (3.3.7) and (4.4.5):

$$
\begin{equation*}
L_{i j}=\frac{\partial v_{i}}{\partial x_{j}}=v_{i I} \frac{\partial N_{I}}{\partial x_{j}} \tag{4.5.1}
\end{equation*}
$$

The above expression can be put into the form of a matrix product if we associate the index $I$ with a column number in $v$ and a row number in $\partial N_{I} / \partial x_{j}$. To simplify the writing of a matrix expression, we define a matrix $\mathcal{B}$ by

$$
\begin{equation*}
\mathcal{B}_{j I}=\frac{\partial N_{I}}{\partial x_{j}} \quad \text { or } \quad \mathrm{B}=\left[\mathcal{B}_{j I}\right]=\left[\partial N_{I} / \partial x_{j}\right] \tag{4.5.2}
\end{equation*}
$$

where $j$ is the row number in the matrix. The velocity gradient can then be expressed in terms of the nodal displacements by (4.5.1) and (4.5.2) by

$$
\begin{equation*}
\left[L_{i j}\right]=\left[v_{i I}\right]\left[\mathcal{B}_{I j}\right]=\left[v_{i I}\right]\left[\mathcal{B}_{j I}\right]^{T} \quad \text { or } \quad \mathbf{L}=\mathbf{v} \mathbf{B}^{T} \tag{4.5.3}
\end{equation*}
$$

so, because of the implicit sum on $I$, the indicial expression corresponds to a matrix product.
We can also often write the expression (4.5.1) without expressing the sum on $I$ in matrix form. The $\mathcal{B}$ matrix is then subdivided into $\mathcal{B}_{I}$ matrices, each associated with node $I$ :

$$
\begin{equation*}
\mathcal{B}=\left[\mathcal{B}_{1}, \mathcal{B}_{2}, \mathcal{B}_{3}, \ldots, \mathcal{B}_{m}\right] \quad \text { where } \mathrm{B}_{I}^{T}=\left\{\mathcal{B}_{j}\right\}_{I}=N_{I, \mathbf{x}} \tag{4.5.3b}
\end{equation*}
$$

For each node $I$, the $\mathcal{B}_{I}$ matrix is a column matrix. Then the expression for the velocity gradient can be written as a sum of tensor products, a product of a column matrix with a row matrix, as shown below

$$
\mathbf{L}=\mathbf{v}_{I} \mathcal{B}_{I}^{T}=\left\{\begin{array}{l}
v_{x I}  \tag{4.5.4}\\
v_{y I}
\end{array}\right\}\left[\begin{array}{ll}
N_{I, x} & N_{I, y}
\end{array}\right]=\left[\begin{array}{ll}
v_{x I} N_{I, x} & v_{x I} N_{I, y} \\
v_{y I} N_{I, x} & v_{y I} N_{I, y}
\end{array}\right]
$$

To put the internal force expression (4.4.11) in matrix form, we first rearrange the terms so that adjacent terms correspond to matrix products. This entails interchanging the row and column number on the internal forces a shown below

$$
\begin{equation*}
\left(f_{i I}^{i n t}\right)^{T}=f_{I i}^{i n t}=\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega=\int_{\Omega} \mathcal{B}_{l j}^{T} \sigma_{j i} d \Omega \tag{4.5.5}
\end{equation*}
$$

The above can be put in the following matrix form

$$
\begin{equation*}
\left[f_{i I}^{i n t}\right]^{T}=\left[f_{I i}^{i n t}\right]=\int_{\Omega}\left[\partial N_{I} / \partial x_{j}\right]\left[\sigma_{j i}\right] d \Omega=\int_{\Omega}\left[\mathcal{B}_{j I}\right]^{T}\left[\sigma_{j i}\right] d \Omega \quad\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\int_{\Omega} \mathcal{B}_{I}^{T} \sigma d \Omega \tag{4.5.6}
\end{equation*}
$$

For example, in two dimensions this gives

$$
\left[\begin{array}{ll}
f_{x I} & f_{y I}
\end{array}\right]^{\text {int }}=\int_{\Omega}\left[N_{I, x} \quad N_{I, y}\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{x y}  \tag{4.5.7}\\
\sigma_{x y} & \sigma_{y y}
\end{array}\right] d \Omega\right.
$$

There are many other ways of converting indicial expressions to matrix form but the above is convenient because it adheres to the convention of treating single index matrices as column matrices, which is customary in the finite element literature. The expression for all nodal forces can be obtained by using the $\mathcal{B}$ matrix as defined in (4.5.3b), which gives

$$
\left(\mathbf{f}^{i n t}\right)^{T}=\int_{\Omega} \mathrm{B}^{T} \sigma d \Omega
$$

4.5.2. Voigt notation. An alternate implementation which is widely used in linear finite element analysis is based on Voigt notation, see Appendix B. Voigt notation is useful for computing tangent stiffness matrices in Newton methods, See Chapter 6. In Voigt notation the stresses and rate-of-deformation are expressed in column vectors, so in two dimensions

$$
\{\mathbf{D}\}^{T}=\left[\begin{array}{lll}
D_{x} & D_{y} & 2 D_{x y}
\end{array}\right] \quad\{\sigma\}^{T}=\left[\begin{array}{lll}
\sigma_{x} & \sigma_{y} & \sigma_{x y} \tag{4.5.11}
\end{array}\right]
$$

We define the $\mathbf{B}_{I}$ matrix so it relates the rate-of-deformation to the nodal velocities by

$$
\begin{equation*}
\{\mathbf{D}\}=\mathbf{B}_{I} \mathbf{v}_{I} \quad\{\delta \mathbf{D}\}=\mathbf{B}_{I} \delta \mathbf{v}_{I} \tag{4.5.12}
\end{equation*}
$$

where the summation convention as usual applies to repeated indices. The elements of the $\mathbf{B}_{I}$ matrix are obtained so as to meet the definition (4.5.12); this is illustrated in the following examples. Note that a matrix is enclosed in brackets only when this is needed to distinguish a matrix from its usual form as a square matrix; matrices and tensors of third order or higher which become square matrices are written simply as boldface.

The expression for the internal force vector can be derived in this notation by using the definition of the virtual internal power in terms of the nodal forces and nodal velocities and in terms of the stresses and rate of deformation, Eq. (4.3.19). Since $\{\mathbf{D}\}^{T}\{\sigma\}$ gives the internal power per unit volume (the column matrices were designed with this in mind), it follows that

$$
\begin{equation*}
\delta \mathscr{P}^{i n t}=\delta \mathbf{v}_{I}^{T} \mathbf{f}_{I}^{i n t}=\int_{\Omega}\{\delta \mathbf{D}\}^{T}\{\sigma\} d \Omega \tag{4.5.13}
\end{equation*}
$$

Substituting (4.5.12) into the above and invoking the arbitrariness of $\{\delta \mathbf{v}\}$ gives

$$
\begin{equation*}
\mathbf{f}_{I}^{i n t}=\int_{\Omega} \mathbf{B}_{I}^{T}\{\sigma\} d \Omega \tag{4.5.14}
\end{equation*}
$$

As will be shown in the examples, Eq. (4.5.14) gives the same expression for the internal nodal forces as Eq. (4.5.6): Eq. (4.5.14) uses the symmetric part of the velocity gradient, whereas the complete velocity gradient has been used in Eq. (4.5.6). However, since the Cauchy stress is symmetric, the two expression are equivalent; this is verified in the following examples.

It is sometimes convenient to place the displacement, velocities and nodal forces for an element or a complete mesh in a single column matrix. We will then use the symbol $\mathbf{d}$ for the column matrix of all nodal displacements, $\mathbf{d}$ for the column matrix of nodal velocities and $\{\mathbf{f}\}$ for the column matrix of nodal forces, i.e.

$$
\mathbf{d}=\left\{\begin{array}{c}
\mathbf{u}_{1}  \tag{4.5.15}\\
\mathbf{u}_{2} \\
\cdot \\
\mathbf{u}_{m}
\end{array}\right\} \quad \dot{\mathbf{d}}=\left\{\begin{array}{c}
\mathbf{v}_{1} \\
\mathbf{v}_{2} \\
\cdot \\
\mathbf{v}_{m}
\end{array}\right\} \quad \mathbf{f}=\left\{\begin{array}{c}
\mathbf{f}_{1} \\
\mathbf{f}_{2} \\
\cdot \\
\mathbf{f}_{m}
\end{array}\right\}
$$

where $m$ is the number of nodes. The correspondence between the two matrices is given by

$$
\begin{equation*}
d_{a}=u_{i I} \text { where } a=(I-1)^{*} n_{S D}+i \tag{4.5.16}
\end{equation*}
$$

Note that we use a different symbol for the column matrix of all nodal displacements and nodal velocities because the symbols $\mathbf{u}$ and $\mathbf{v}$ refer to the displacement and velocity vector fields in the continuum mechanics description.

In this notation, we can write the counterpart of Eq. (4.5.12) as

$$
\begin{equation*}
\{\mathbf{D}\}=\mathbf{B} \dot{\mathbf{d}} \quad \text { where } \mathbf{B}=\left[\mathbf{B}_{1}, \mathbf{B}_{2}, \ldots, \mathbf{B}_{m}\right] \tag{4.5.18}
\end{equation*}
$$

where the brackets around $\mathbf{D}$ indicate that the tensor is stored in colum matrix form; we do not put brackets around $\mathbf{B}$ because this is always a rectangular matrix. The nodal forces are given by the counterpart of Eq. (4.5.14):

$$
\begin{equation*}
\{\mathbf{f}\}^{i n t}=\int_{\Omega} \mathbf{B}^{T}\{\sigma\} d \Omega \tag{4.5.19}
\end{equation*}
$$

Often we omit the brackets on the nodal force, since the presence of a single term in Voigt notation always indicates that the entire equation is in Voigt notation. The Voigt form can also be obtained by rewriting (4.5.5) as

$$
\begin{equation*}
f_{I r}^{i n t}=\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \delta_{r i} \sigma_{j i} d \Omega \tag{4.5.20}
\end{equation*}
$$

Then defining the $\mathbf{B}$ matrix by

$$
\begin{equation*}
B_{i j I r}=\frac{\partial N_{I}}{\partial x_{j}} \delta_{r i} \tag{4.5.21}
\end{equation*}
$$

and converting the indices $(i, j)$ by the kinematic Voigt rule to a and the indices (I.r) by the matrixcolumn vector rule gives

$$
\begin{equation*}
f_{a}^{i n t}=\int_{\Omega} B_{b a}^{T} \sigma_{b} d \Omega \quad \text { or } \quad \mathbf{f}^{\text {int }}=\int_{\Omega} \mathbf{B}^{\mathbf{T}}\{\sigma\} \sigma d \Omega \tag{4.5.22}
\end{equation*}
$$

More detail and techniques for translating indicial notation to Voigt notation can be found in Appendix B.
4.5.4. Numerical Quadrature. The integrals for the nodal forces, mass matrix and other element matrices can generally not be evaluated in closed form, and are instead integrated numerically (often called numerical quadrature). The most widely used procedure for numerical integration in finite elements is Gauss quadrature. The Gauss quadrature formulas are, see for example Dhatt and Touzot (1984, p.240), Hughes (1977, p. 137)

$$
\begin{equation*}
\int_{-1}^{1} f(\xi) d \xi=\sum_{Q=1}^{n_{Q}} w_{Q} f\left(\xi_{Q}\right) \tag{4.5.24}
\end{equation*}
$$

where the weights $w_{Q}$ and coordinates $\xi_{Q}$ of the $n_{Q}$ quadrature points are available in tables; a short table is given in Appendix C. Equation (4.5.24) integrates $f(\xi)$ exactly if it is a polynomial of order $m \leq 2 n_{Q}-1$. Equation (4.5.24) is tailored to quadrature over the parent element domains, since it is over the interval $[-1,1]$.

To integrate over a two dimensional element, the procedure is repeated over the second direction, yielding the following

$$
\begin{equation*}
\int_{\Delta} f(\xi) d \Delta=\int_{-1-1}^{1} \int_{-1}^{1} f(\xi, \eta) d \xi d \eta=\sum_{Q_{1}=1}^{n_{Q_{1}}} \sum_{Q_{2}=1}^{n_{Q_{2}}} w_{Q_{1}} w_{Q_{2}} f\left(\xi_{Q_{1}}, \eta_{Q_{2}}\right) \tag{4.5.25}
\end{equation*}
$$

In three dimensions, the Gauss quadrature formula is

$$
\begin{equation*}
\int_{\Delta} f(\xi) d \Delta=\int_{-1-1}^{1} \int_{1}^{1} \int_{1}^{1} f(\xi) d \xi d \eta d \zeta=\sum_{Q_{1}=1}^{n_{Q_{1}}} \sum_{Q_{2}=1}^{n_{Q_{2}}} \sum_{Q_{3}=1}^{n_{Q_{3}}} w_{Q_{1}} w_{Q_{2}} w_{Q_{3}} f\left(\xi_{Q_{1}}, \eta_{Q_{2}}, \zeta_{Q_{3}}\right) \tag{4.5.26}
\end{equation*}
$$

For example, in integrating the expression for the internal nodal forces over the biunit square parent element in two dimensions, we use

$$
\begin{align*}
\mathbf{f}^{i n t} & =\int_{\Delta} \mathbf{B}^{T}\{\sigma\} J_{\xi} d \Delta=\int_{-1-1}^{1} \int \mathbf{B}^{T}\{\sigma\} J_{\xi} d \xi d \eta  \tag{4.5.27}\\
& =\sum_{Q_{1}=1}^{n_{Q_{1}}} \sum_{Q_{2}=1}^{n_{Q_{2}}} w_{Q_{1}} w_{Q_{2}} \mathbf{B}^{T}\left(\xi_{Q_{1}}, \eta_{Q_{2}}\right)\left\{\sigma\left(\xi_{Q_{1}}, \eta_{Q_{2}}\right)\right\} J_{\xi}\left(\xi_{Q_{1}}, \eta_{Q_{2}}\right)
\end{align*}
$$

To simplify the notation for multi-dimensional quadrature, we often combine the weights into a single weight and write the quadrature formula in any dimension as

$$
\begin{equation*}
\int_{\Delta} f(\xi) d \Delta=\sum_{Q} \bar{w}_{Q} f\left(\xi_{Q}\right) \tag{4.5.28}
\end{equation*}
$$

where $\bar{w}_{Q}$ is a products of the weights for one-dimensional quadrature $w_{Q}$.
The number of quadrature points used in nonlinear analysis is generally based on the same rules as for linear analysis; the number of quadrature points is chosen to exactly integrate the nodal internal forces for a regular element. A regular form of an element is one that can be obtained by only stretching but not shearing the parent element; for example, a rectangle for two-dimensional isoparametric elements. To choose the number of quadrature points for the internal nodal forces for a 4 -node quadrilateral, we use the following argument. The rate-of-deformation and the $\mathbf{B}$ matrix are linear in this element since the velocities are bilinear. If the stress is linearly related to the rate-of-deformation, it will vary linearly within the quadrilateral element. The integrand for the internal nodal forces is approximately quadratic, since it is a product of the $\mathbf{B}$-matrix and the stresses. By the above rule for Gauss quadrature, two quadrature points are then needed in each direction for exact quadrature of a quadratic function, so $2 \times 2$ quadrature on a quadrilateral integrates the internal nodal forces exactly on a regular element. Quadrature formulas which integrate the nodal internal forces almost exactly for a linear constitutive equation are called full quadrature.

Gauss quadrature is very powerful for smooth functions which are polynomials or nearly polynomials. In linear finite element analysis, the integrand in the expression for the stiffness matrix consists of polynomials for rectangular elements and is smooth and nearly a polynomial for isoparametric elements. In nonlinear analysis, the integrand is not always smooth. For example, for an elastic-plastic material, the stress may have a discontinuous derivative in space at the surface separating elastic and plastic material. Even if the stress-strain law is smooth for an elastic-plastic material, the derivative of the stress with respect to the strain changes drastically when the response changes from elastic to plastic, so the effect is the same. Therefore, the errors in Gauss quadrature of an element that contains an elastic-plastic interface are likely to be large. However, higher order quadrature is not recommended for circumventing these errors, since it often leads to stiff behavior or locking of elements.
4.5.5. Selective-Reduced Integration. For incompressible or nearly incompressible materials, full quadrature of the nodal internal forces may cause an element to lock, i.e. the displacements are very small and do not converge or converge very slowly. The easiest way to circumvent this difficulty is to use selective-reduced integration.

In selective-reduced integration, the pressure is underintegrated, whereas the remainder of the stress matrix is fully integrated. For this purpose, the stress tensor is decomposed into the hydrostatic component or pressure $p$, which is the trace of the stress tensor

$$
\begin{equation*}
\sigma_{i j}^{d i l}=p \delta_{i j}=\frac{1}{3} \sigma_{k k} \delta_{i j} \tag{4.5.29a}
\end{equation*}
$$

and the deviatoric components:

$$
\begin{equation*}
\sigma_{i j}^{d e v}=\sigma_{i j}-p \delta_{i j} \tag{4.5.29b}
\end{equation*}
$$

The rate-of-deformation is similarly split into dilatational and deviatoric components which are defined by

$$
\begin{equation*}
D_{i j}^{d e v}=D_{i j}-\frac{1}{3} D_{k k} \delta_{i j} \quad D_{i j}^{d i l}=\frac{1}{3} D_{k k} \delta_{i j} \tag{4.5.30}
\end{equation*}
$$

It is noted that the dilatational and deviatoric components are orthogonal to each other so that the total virtual internal power as defined in Eq. (4.3.19) becomes:

$$
\begin{equation*}
\delta P^{i n t}=\int_{\Omega} \delta D_{i j} \sigma_{i j} d \Omega=\frac{1}{3} \int_{\Omega} \delta D_{i i} p d \Omega+\int_{\Omega} \delta D_{i j}^{d e v} \sigma_{i j}^{d e v} d \Omega \tag{4.5.31}
\end{equation*}
$$

After expressing the rate-of-deformation in terms of the shape functions by (4.4.7b) and (4.5.30), the dilatational and deviatoric integrands become:

$$
\begin{equation*}
\delta D_{i i} p=\delta v_{i l} N_{I, i} p \tag{4.5.32a}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta D_{i j}^{d e v} \sigma_{i j}^{d e v}=\frac{1}{2}\left(N_{I, j} \delta v_{i I}+N_{I, i} \delta v_{j I}\right) \sigma_{i j}^{d e v}-\frac{1}{3} N_{I, k} \delta v_{k l} \delta_{i j} \sigma_{i j}^{d e v} \tag{4.5.32b}
\end{equation*}
$$

Using the symmetry of $\sigma_{i j}^{d e v}$ and the fact that the trace of $\sigma_{i j}^{d e v}$ vanishes, the deviatoric integrand simplifies to:

$$
\begin{equation*}
\delta D_{i j}^{d e v} \sigma_{i j}^{d e v}=\delta v_{i I} N_{I, j} \sigma_{i j}^{d e v} \tag{4.5.33}
\end{equation*}
$$

Selective-reduced integration consists of full integration on the deviatoric term and reduced integration on the dilatational term on $\delta P^{i n t}$. Thus, for a four-node quadrilateral, selective-reduced integration gives:

$$
\begin{equation*}
\delta \mathcal{P}^{i n t}=\delta v_{i I}\left[\frac{1}{3} J_{\xi}(\mathbf{0}) N_{I, i}(\mathbf{0}) p(\mathbf{0})+\sum_{Q=1}^{4} \bar{w}_{Q} J_{\xi}\left(\xi_{Q}\right) N_{I, j}\left(\xi_{Q}\right) \sigma_{j i}^{d e v}\left(\xi_{Q}\right)\right] \tag{4.3.34a}
\end{equation*}
$$

Hence the selective-reduced integration internal force expression is:

$$
\begin{equation*}
\left(f_{i I}^{i n t}\right)^{T}=f_{I i}^{i n t}=\left[\frac{1}{3} J_{\xi}(\mathbf{0}) N_{I, i}(\mathbf{0}) p(\mathbf{0})+\sum_{Q=1}^{4} \bar{w}_{Q} J_{\xi}\left(\xi_{Q}\right) N_{I, j}\left(\xi_{Q}\right) \sigma_{j i}^{d e v}\left(\xi_{Q}\right)\right] \tag{4.3.34b}
\end{equation*}
$$

where, as indicated in the above, the single quadrature point for the reduced quadrature is the centroid of the parent element. The deviatoric part is integrated by full quadrature using two points in each direction, for a total of four quadrature points; this is called $2 \times 2$ quadrature. This scheme is similar to the scheme used in linear analysis of incompressible materials. Selective-reduced schemes for other elements can be developed by similarly modifying selective-reduced integration schemes given linear finite element texts; see Hughes(1979) for selective-reduced integration procedures for linear problems.
4.5.6. Element Force and Matrix Transformations. Often element nodal forces and element matrices must be expressed which in terms of alternate degrees of freedom, i.e. for a
different set of nodal displacements. In the following, transformations are developed for nodal forces and element matrices.

Consider an element or assemblage of elements with nodal displacements $\hat{\mathbf{d}}$. We wish to express the nodal forces for the nodal displacements $\mathbf{d}$ which are related to $\hat{\mathbf{d}}$ by

$$
\begin{equation*}
\frac{d \hat{\mathbf{d}}}{d t}=\mathbf{T} \frac{d \mathbf{d}}{d t} \quad \text { or } \quad \delta \hat{\mathbf{d}}=\mathbf{T} \delta \mathbf{d} \tag{4.5.35}
\end{equation*}
$$

The nodal forces associated with $\mathbf{d}$ are then given by

$$
\begin{equation*}
\mathbf{f}=\mathbf{T}^{T} \hat{\mathbf{f}} \tag{4.5.36}
\end{equation*}
$$

This transformation holds because the nodal foces and velocities are assumed to be conjugate in power, see Section 2.4.2. It is proven as follows. An increment of work is given by

$$
\begin{equation*}
\delta W=\delta \mathbf{d}^{T} \mathbf{f}=\delta \hat{\mathbf{d}}^{T} \mathbf{f} \quad \forall \delta \mathbf{d} \tag{4.5.37}
\end{equation*}
$$

Either set of nodal forces and virtual displacements must give an increment in work, since work is a scalar independent of the coordinate system or choice of generalized displacements. Substituting (4.5.35) into (4.5.37) gives

$$
\begin{equation*}
\delta \mathbf{d}^{T} \mathbf{f}=\delta \mathbf{d}^{T} \mathbf{T}^{T} \hat{\mathbf{f}} \quad \forall \delta \mathbf{d} \tag{4.5.38}
\end{equation*}
$$

Since (4.5.38) holds for all $\delta \mathbf{d}$, Eq.(4.5.36) follows.
The mass matrix can be transformed similarly. We first consider the case where $\mathbf{T}$ is independent of time. then the mass matrix for the two set of degrees of freedom is related by

$$
\begin{equation*}
\mathbf{M}=\mathbf{T}^{T} \hat{\mathbf{M}} \mathbf{T} \tag{4.5.39}
\end{equation*}
$$

This is shown as follows. By Eq.(4.5.36)

$$
\begin{equation*}
\mathbf{f}^{\text {inert }}=\mathbf{T}^{T} \mathbf{f}^{\text {inert }} \tag{4.5.40}
\end{equation*}
$$

and using (4.4.18) for the two sets of degrees of freedom, we have

$$
\begin{equation*}
\mathbf{M} \dot{\mathbf{v}}=\mathbf{T}^{T} \hat{\mathbf{M}} \dot{\hat{\mathbf{v}}} \tag{4.5.41}
\end{equation*}
$$

If $\mathbf{T}$ is independent of time, from (4.5.35), $\dot{\hat{\mathbf{v}}}=\mathbf{T} \dot{\mathbf{v}}$, and substituting this into the above and using the fact that this must hold for the arbitrary nodal accelerations, we obtain (4.5.39). If the $\mathbf{T}$ matrix is time dependent, then $\dot{\hat{\mathbf{v}}}=\mathbf{T} \dot{\mathbf{v}}+\dot{\mathbf{T}} \mathbf{v}$, so

$$
\begin{equation*}
\mathbf{f}^{i n e r t}=\mathbf{T}^{T} \hat{\mathbf{M}} \ddot{\mathbf{d}}+\mathbf{T}^{T} \hat{\mathbf{M}} \dot{\mathbf{T}} \dot{\mathbf{d}} \tag{4.5.42}
\end{equation*}
$$

The matrix $\dot{\mathbf{T}}$ usually depends on the nodal velocities, so in this case terms which are not linear in the velocities occur in the equations of motion.

A transformation similar to (4.3.39) holds for the linear stiffness matrix and the tangent stiffness discussed in Chapter 6:

$$
\begin{equation*}
\mathbf{K}=\mathbf{T}^{T} \hat{\mathbf{K}} \mathbf{T}, \quad \mathbf{K}^{t a n}=\mathbf{T}^{T} \hat{\mathbf{K}}^{t a n} \mathbf{T} \tag{4.5.43}
\end{equation*}
$$

These transformations enable us to evaluate element matrices in coordinate systems which simplify the procedure as in example 4.6. They are also useful for treating slave nodes, see example 4.5

Example 4.1. Triangular 3-node element. The triangular element will be developed using triangular coordinates (also called area coordinates and barycentric coordinates). The element is shown in Figure 4.2. It is a 3-node element with a linear displacement field; the thickness of the element is $a$. The nodes are numbered counterclockwise in the parent element, and they must be numbered counterclockwise in the initial configuration, otherwise the determinant of the map between the initial and parent domains will be negative.


Fig. 4.2. Triangular element showing node numbers and the mappings of the initial and current configurations to the parent element.

The shape functions for the linear displacement triangle are the triangular coordinates, so $N_{I}=\xi_{I}$. The spatial coordinates are expressed in terms of the triangular coordinates $\xi_{I}$ by

$$
\left\{\begin{array}{l}
x  \tag{E4.1.2}\\
y \\
1
\end{array}\right\}=\left[\begin{array}{ccc}
x_{1} & x_{2} & x_{3} \\
y_{1} & y_{2} & y_{3} \\
1 & 1 & 1
\end{array}\right]\left[\begin{array}{l}
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right\}
$$

where we have appended the condition that the sum of the triangular element coordinates is one. The inverse of Eq. (E4.1.2) is given by

$$
\left\{\begin{array}{l}
\xi_{1}  \tag{E4.1.4a}\\
\xi_{2} \\
\xi_{3}
\end{array}\right\}=\frac{1}{2 A}\left[\begin{array}{lll}
y_{23} & x_{32} & x_{2} y_{3}-x_{3} y_{2} \\
y_{31} & x_{13} & x_{3} y_{1}-x_{1} y_{3} \\
y_{12} & x_{21} & x_{1} y_{2}-x_{2} y_{1}
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
1
\end{array}\right\}
$$

where we have used the notation

$$
\begin{equation*}
x_{I J}=x_{I}-x_{J} \quad y_{I J}=y_{I}-y_{J} \tag{E4.1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
2 A=x_{32} y_{12}-x_{12} y_{32} \tag{E4.1.4b}
\end{equation*}
$$

where $A$ is the current area of the element. As can be seen from the above, in the triangular threenode element, the parent to current map (E4.1.2) can be inverted explicitly. This unusual circumstance is due to the fact that the map for this element is linear. However, the parent to current map is nonlinear for most other elements, so for most elements it cannot be inverted.

The derivatives of the shape functions can be determined directly from (E4.1.4a) by inspection:

$$
\left[N_{I, j}\right]=\left[\xi_{I, j}\right]=\left[\begin{array}{ll}
\xi_{1, x} & \xi_{1, y}  \tag{E4.1.5}\\
\xi_{2, x} & \xi_{2, y} \\
\xi_{3, x} & \xi_{3, y}
\end{array}\right]=\frac{1}{2 A}\left[\begin{array}{cc}
y_{23} & x_{32} \\
y_{31} & x_{13} \\
y_{12} & x_{21}
\end{array}\right]
$$

We can obtain the map between the parent element and the initial configuration by writing Eq. (E4.1.2) at time $t=0$, which gives

$$
\left\{\begin{array}{c}
X  \tag{E4.1.6}\\
Y \\
1
\end{array}\right\}=\left[\begin{array}{ccc}
X_{1} & X_{2} & X_{3} \\
Y_{1} & Y_{2} & Y_{3} \\
1 & 1 & 1
\end{array}\right]\left\{\begin{array}{l}
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right\}
$$

The inverse of this relation is identical to (E4.1.4) except that it is in terms of the initial coordinates

$$
\left\{\begin{array}{l}
\xi_{1}  \tag{E4.1.7a}\\
\xi_{2} \\
\xi_{3}
\end{array}\right\}=\frac{1}{2 A_{0}}\left[\begin{array}{lll}
Y_{23} & X_{32} & X_{2} Y_{3}-X_{3} Y_{2} \\
Y_{31} & X_{13} & X_{3} Y_{1}-X_{1} Y_{3} \\
Y_{12} & X_{21} & X_{1} Y_{2}-X_{2} Y_{1}
\end{array}\right]\left\{\begin{array}{c}
x \\
y \\
1
\end{array}\right\}
$$

$$
\begin{equation*}
2 A_{0}=X_{32} Y_{12}-X_{12} Y_{32} \tag{E4.1.7b}
\end{equation*}
$$

where $A_{0}$ is the initial area of the element.
Voigt Notation. We first develop the element equations in Voigt notation, which should be familiar to those who have studied linear finite elements. Those who like more condensed matrix notation can skip directly to that form. In Voigt notation, the displacement field is often written in terms of triangular coordinates as

$$
\left\{\begin{array}{l}
u_{x}  \tag{E4.1.8}\\
u_{y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\xi_{1} & 0 & \xi_{2} & 0 & \xi_{3} & 0 \\
0 & \xi_{1} & 0 & \xi_{2} & 0 & \xi_{3}
\end{array}\right] \mathbf{d}=\mathbf{N} \mathbf{d}
$$

where $\mathbf{d}$ is the column matrix of nodal displacements, which is given by

$$
\begin{equation*}
\mathbf{d}^{T}=\left[u_{x 1}, u_{y 1}, u_{x 2}, u_{y 2}, u_{x 3}, u_{y 3}\right] \tag{E4.1.9}
\end{equation*}
$$

We will generally not use this form, since it includes many zeroes and write the displacement in a form similar to (E4.4.1). The velocities are obtained by taking the material time derivatives of the displacements, giving

$$
\begin{align*}
& \left\{\begin{array}{l}
v_{x} \\
v_{y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\xi_{1} & 0 & \xi_{2} & 0 & \xi_{3} & 0 \\
0 & \xi_{1} & 0 & \xi_{2} & 0 & \xi_{3}
\end{array}\right] \dot{\mathbf{d}}  \tag{E4.1.10}\\
& \dot{\mathbf{d}}^{T}=\left[v_{x 1}, v_{y 1}, v_{x 2}, v_{y 2}, v_{x 3}, v_{y 3}\right] \tag{E4.1.11}
\end{align*}
$$

The nodal velocities and nodal forces of the element are shown in Fig. 4.3.


Fig. 4.3. Triangular element showing the nodal force and velocity components.

The rate-of-deformation and stress column matrices in Voigt form are

$$
\{\mathbf{D}\}=\left\{\begin{array}{c}
D_{x x}  \tag{E4.1.12}\\
D_{y y} \\
2 D_{x y}
\end{array}\right\} \quad\{\sigma\}=\left\{\begin{array}{c}
\sigma_{x x} \\
\sigma_{y y} \\
\sigma_{x y}
\end{array}\right\}
$$

where the factor of 2 on the shear velocity strain is needed in Voigt notation; see the Appendix B. Only the in-plane stresses are needed in either plane stress or plain strain, since $\sigma_{z z}=0$ in plane stress whereas $D_{z z}=0$ in plane strain, so $D_{z z} \sigma_{z z}$ makes no contribution to the power in either case. The transverse shear stresses, $\sigma_{x z}$ and $\sigma_{y z}$, and the corresponding components of the rate-of-deformation, $D_{x z}$ and $D_{y z}$, vanish in both plane stress and plane strain problems.

By the definition of the rate-of-deformation, Equations (3.3.10) and the velocity approximation, we have

$$
\begin{align*}
& D_{x x}=\frac{\partial v_{x}}{\partial x}=\frac{\partial N_{I}}{\partial x} v_{I x} \\
& D_{y y}=\frac{\partial v_{y}}{\partial y}=\frac{\partial N_{I}}{\partial y} v_{I y}  \tag{E4.1.13}\\
& 2 D_{x y}=\frac{\partial v_{x}}{\partial y}+\frac{\partial v_{y}}{\partial x}=\frac{\partial N_{I}}{\partial y} v_{I x}+\frac{\partial N_{I}}{\partial x} v_{I y}
\end{align*}
$$

In Voigt notation, the $\mathbf{B}$ matrix is developed so it relates the rate-of-deformation to the nodal velocities by $\{\mathbf{D}\}=\mathbf{B} \dot{\mathbf{d}}$, so using (E4.1.13) and the formulas for the derivatives of the triangular coordinates (E4.1.5), we have

$$
\mathbf{B}_{I}=\left[\begin{array}{cc}
N_{I, x} & 0  \tag{E4.1.14}\\
0 & N_{I, y} \\
N_{I, y} & N_{I, x}
\end{array}\right] \quad[\mathbf{B}]=\left[\begin{array}{lll}
\mathbf{B}_{1} & \mathbf{B}_{2} & \mathbf{B}_{3}
\end{array}\right]=\frac{1}{2 A}\left[\begin{array}{cccccc}
y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\
0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\
x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12}
\end{array}\right]
$$

The internal nodal forces are then given by (4.5.14):

$$
\left\{\begin{array}{l}
f_{x 1}  \tag{E4.1.15}\\
f_{y 1} \\
f_{x 2} \\
f_{y 2} \\
f_{x 3} \\
f_{y 3}
\end{array}\right\}=\int_{\Omega} \mathbf{B}^{T}\{\sigma\} d \Omega=\int_{\Omega} \frac{a}{2 A}\left[\begin{array}{ccc}
y_{23} & 0 & x_{32} \\
0 & x_{32} & y_{23} \\
y_{31} & 0 & x_{13} \\
0 & x_{13} & y_{31} \\
y_{12} & 0 & x_{21} \\
0 & x_{21} & y_{12}
\end{array}\right]\left\{\left\{\begin{array}{c}
\sigma_{x x} \\
\sigma_{y y} \\
\sigma_{x y}
\end{array}\right\} d A\right.
$$

where $a$ is the thickness and we have used $d \Omega=a d A$; if we assume that the stresses and thickness $a$ are constant in the element, we obtain

$$
\left\{\begin{array}{l}
f_{x 1}  \tag{E4.1.16}\\
f_{y 1} \\
f_{x 2} \\
f_{y 2} \\
f_{x 3} \\
f_{y 3}
\end{array}\right\}=\frac{a}{2}\left[\begin{array}{ccc}
y_{23} & 0 & x_{32} \\
0 & x_{32} & y_{23} \\
y_{31} & 0 & x_{13} \\
0 & x_{13} & y_{31} \\
y_{12} & 0 & x_{21} \\
0 & x_{21} & y_{12}
\end{array}\right]\left\{\begin{array}{l}
\sigma_{x x} \\
\sigma_{y y} \\
\sigma_{x y}
\end{array}\right\}
$$

In the 3-node triangle, the stresses are sometimes not constant within the element; for example, when thermal stresses are included for a linear temperature field, the stresses are linear. In this case, or when the thickness $a$ varies in the element, one-point quadrature is usually adequate. Onepoint quadrature is equivalent to (E4.1.16) with the stresses and thickness evaluated at the centroid of the element.

Matrix Form based on Indicial Notation. In the following, the expressions for the element are developed using a direct translation of the indicial expression to matrix form. The equations are more compact but not in the form commonly seen in linear finite element analysis.

Rate-of-Deformation. The velocity gradient is given by a matrix form of (4.4.7)

$$
\begin{align*}
\mathbf{L} & =\left[L_{i j}\right]=\left[v_{i I}\right]\left[N_{I, j}\right]=\left[\begin{array}{lll}
v_{x 1} & v_{x 2} & v_{x 3} \\
v_{y 1} & v_{y 2} & v_{y 3}
\end{array}\right] \frac{1}{2 A}\left[\begin{array}{ll}
y_{23} & x_{32} \\
y_{31} & x_{13} \\
y_{12} & x_{21}
\end{array}\right]= \\
& =\frac{1}{2 A}\left[\begin{array}{ll}
y_{23} v_{x 1}+y_{31} v_{x 2}+y_{12} v_{x 3} & x_{32} v_{x 1}+x_{13} v_{x 2}+x_{21} v_{x 3} \\
y_{23} v_{y 1}+y_{31} v_{y 2}+y_{12} v_{y 3} & x_{32} v_{y 1}+x_{13} v_{y 2}+x_{21} v_{y 3}
\end{array}\right] \tag{E4.1.19}
\end{align*}
$$

The rate-of-deformation is obtained from the above by (3.3.10):

$$
\begin{equation*}
\mathbf{D}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{T}\right) \tag{E4.1.20}
\end{equation*}
$$

As can be seen from (E4.1.19) and (E4.1.20), the rate-of-deformation is constant in the element; the terms $x_{I J}$ and $y_{I J}$ are differences in nodal coordinates, not functions of spatial coordinates.

Internal Nodal Forces. The internal forces are given by (4.5.10) using (E4.1.5) for the derivatives of the shape functions:

$$
\mathbf{f}_{\text {int }}^{T}=\left[f_{I i}\right]^{\text {int }}=\left[\begin{array}{ll}
f_{1 x} & f_{1 y} \\
f_{2 x} & f_{2 y} \\
f_{3 x} & f_{3 y}
\end{array}\right]^{\text {int }}=\int_{\Omega}\left[N_{I, j}\right]\left[\sigma_{j i}\right] d \Omega=\int_{A} \frac{1}{2 A}\left[\begin{array}{ll}
y_{23} & x_{32} \\
y_{31} & x_{13} \\
y_{12} & x_{21}
\end{array}\right]\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y y}
\end{array}\right] a d A(\mathrm{E} 4.1 .21)
$$

where $a$ is the thickness. If the stresses and thickness are constant within the element, the integrand is constant and the integral can be evaluated by multiplying the integrand by the volume $a A$, giving

$$
\mathbf{f}_{i n t}^{T}=\frac{a}{2}\left[\begin{array}{ll}
y_{23} & x_{32}  \tag{E4.1.22}\\
y_{31} & x_{13} \\
y_{12} & x_{21}
\end{array}\right]\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y y}
\end{array}\right]=\frac{a}{2}\left[\begin{array}{ll}
y_{23} \sigma_{x x}+x_{32} \sigma_{x y} & y_{23} \sigma_{x y}+x_{32} \sigma_{y y} \\
y_{31} \sigma_{x x}+x_{13} \sigma_{x y} & y_{31} \sigma_{x y}+x_{13} \sigma_{y y} \\
y_{12} \sigma_{x x}+x_{21} \sigma_{x y} & y_{12} \sigma_{x y}+x_{21} \sigma_{y y}
\end{array}\right]
$$

This expression gives the same result as Eq. (E4.1.16). It is easy to show that the sums of each of the components of the nodal forces vanish, i.e. the element is in equilibrium. Comparing (E4.1.21) with (E4.1.16), we see that the matrix form of the indicial expression involves fewer multiplications. In evaluating the Voigt form (E4.1.16) involves many multiplications with zero, which slows computations, particularly in the three-dimensional counterparts of these equations. However, the matrix indicial form is difficult to extend to the computation of stiffness matrices, so as will be seen in Chapter 6, the Voigt form is indispensible when stiffness matrices are needed.

Mass Matrix. The mass matrix is evaluated in the undeformed configuration by (4.4.52). The mass matrix is given by

$$
\begin{equation*}
\tilde{M}_{I J}=\int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0}=\int_{\Delta} a_{0} \rho_{0} \xi_{I} \xi_{J} J_{\xi}^{0} d \Delta \tag{E4.1.23}
\end{equation*}
$$

where we have used $d \Omega_{0}=a_{0} J_{\xi}^{0} d \Delta$; the quadrature in the far right expression is over the parent element domain. Putting this in matrix form gives

$$
\tilde{\mathbf{M}}=\int_{\Delta} a_{0} \rho_{0}\left[\begin{array}{l}
\xi_{1}  \tag{E4.1.24}\\
\xi_{2} \\
\xi_{3}
\end{array}\right]\left[\begin{array}{lll}
\xi_{1} & \xi_{2} & \xi_{3}
\end{array}\right] J_{\xi}^{0} d \Delta
$$

where the element Jacobian determinant for the initial configuration of the triangular element is given by $J_{\xi}^{0}=2 A_{0}$, where $A_{0}$ is the initial area. Using the quadrature rule for triangular coordinates, the consistent mass matrix is:

$$
\left.\tilde{\mathbf{M}}=\frac{\rho_{0} A_{0} a_{0}}{12} \left\lvert\, \begin{array}{lll}
2 & 1 & 1  \tag{E4.1.25}\\
1 & 2 & 1 \\
1 & 1 & 2
\end{array}\right.\right]
$$

The mass matrix can be expanded to full size by using Eq. (4.4.46), $M_{i J j J}=\delta_{i j} \tilde{M}_{I J}$ and then using the rule of Eq. (1.4.26), which gives

$$
\mathbf{M}=\frac{\rho_{0} A_{0} a_{0}}{12}\left[\begin{array}{llllll}
2 & 0 & 1 & 0 & 1 & 0  \tag{E4.1.26}\\
0 & 2 & 0 & 1 & 0 & 1 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
1 & 0 & 1 & 0 & 2 & 0 \\
0 & 1 & 0 & 1 & 0 & 2
\end{array}\right]
$$

The diagonal or lumped mass matrix can be obtained by the row-sum technique, giving

$$
\tilde{\mathbf{M}}=\frac{\rho_{0} A_{0} a_{0}}{3}\left[\begin{array}{lll}
1 & 0 & 0  \tag{E4.1.27}\\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

This matrix could also be obtained by simply assigning one third of the mass of the element to each of the nodes.

External Nodal Forces. To evaluate the external forces, an interpolation of these forces is needed. Let the body forces be approximated by linear interpolants expressed in terms of the triangular coordinates as

$$
\left\{\begin{array}{l}
b_{x}  \tag{E4.1.28}\\
b_{y}
\end{array}\right\}=\left[\begin{array}{lll}
b_{x 1} & b_{x 2} & b_{x 3} \\
b_{y 1} & b_{y 2} & b_{y 3}
\end{array}\right]\left[\begin{array}{l}
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right\}
$$

Interpretation of Equation (4.4.13) in matrix form then gives

$$
\mathbf{f}_{e x t}^{T}=\left[\begin{array}{lll}
f_{x 1} & f_{x 2} & f_{x 3}  \tag{E4.1.29}\\
f_{y 1} & f_{y 2} & f_{y 3}
\end{array}\right]^{e x t}=\left[\begin{array}{lll}
b_{x 1} & b_{x 2} & b_{x 3} \\
b_{y 1} & b_{y 2} & b_{y 3}
\end{array}\right]\left[\begin{array}{l}
\xi_{1} \\
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right]\left[\begin{array}{lll}
\xi_{1} & \xi_{2} & \xi_{3}
\end{array}\right] \rho a d A
$$

Using the integration rule for triangular coordinates with the thickness and density considered constant then gives

$$
\mathbf{f}_{e x t}^{T}=\frac{\rho A a}{12}\left[\begin{array}{lll}
b_{x 1} & b_{x 2} & b_{x 3}  \tag{E4.1.30}\\
b_{y 1} & b_{y 2} & b_{y 3}
\end{array}\right]\left[\begin{array}{lll}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{array}\right]
$$

To illustrate the formula for the computation of the external forces due to a prescribed traction, consider component $i$ of the traction to be prescribed between nodes 1 and 2. If we approximate the traction by a linear interpolation, then

$$
\begin{equation*}
\bar{t}_{i}=\bar{t}_{i 1} \xi_{1}+\bar{t}_{i} \xi_{2} \tag{E4.1.31}
\end{equation*}
$$

The external nodal forces are given by Eq. (4.4.13). We develop a row of the matrix:

$$
\left[\begin{array}{lll}
f_{i 1} & f_{i 2} & f_{i 3}
\end{array}\right]^{\text {ext }}=\int_{\Gamma_{12}} \bar{t}_{i} N_{I} d \Gamma=\int_{0}^{1}\left(\bar{t}_{i 1} \xi_{1}+\bar{t}_{i 2} \xi_{2}\right)\left[\begin{array}{lll}
\xi_{1} & \xi_{2} & \xi_{3} \tag{E4.1.32}
\end{array}\right] a \ell_{12} d \xi_{1}
$$

where we have used $d s=\ell_{12} d \xi_{1} ; \ell_{12}$ is the current length of the side connecting nodes 1 and 2 . Along this side, $\xi_{2}=1-\xi_{1}, \xi_{3}=0$ and evaluation of the integral in (E4.1.32) gives

$$
\left[\begin{array}{lll}
f_{i 1} & f_{i 2} & f_{i 3}
\end{array}\right]^{e x t}=\frac{a \ell_{12}}{6}\left[\begin{array}{lll}
2 \bar{t}_{i 1}+\bar{t}_{i 2} & \bar{t}_{i 1}+2 \bar{t}_{i 2} & 0 \tag{E4.1.33}
\end{array}\right]
$$

The nodal forces are nonzero only on the nodes of the side to which the traction is applied. This equation holds for an arbitrary local coordinate system. For an applied pressure, the above would be evaluated with a local coordinate system with one coordinate along the element edge.

Example 4.2. Quadrilateral Element and other Isoparametric 2D Elements. Develop the expressions for the deformation gradient, the rate-of-deformation, the nodal forces and the mass matrix for two-dimensional isoparametric elements. Detailed expressions are given for the 4-node quadrilateral. Expressions for the nodal internal forces are given in matrix form.


Fig. 4.4. Quadrilateral element in current and initial configurations and the parent domain.
Shape Functions and Nodal Variables. The element shape functions are expressed in terms of the element coordinates $(\xi, \eta)$. At any time $t$, the spatial coordinates can be expressed in terms of the shape functions and nodal coordinates by

$$
\left\{\begin{array}{l}
x(\xi, t)  \tag{E4.2.1}\\
y(\xi, t)
\end{array}\right\}=N_{I}(\xi)\left\{\begin{array}{l}
x_{I}(t) \\
y_{I}(t)
\end{array}\right\}, \quad \xi=\left\{\begin{array}{l}
\xi \\
\eta
\end{array}\right\}
$$

For the quadrilateral, the isoparametric shape functions are

$$
\begin{equation*}
N_{I}(\xi)=\frac{1}{4}\left(1+\xi_{I} \xi\right)\left(1+\eta_{I} \eta\right) \tag{E4.2.2}
\end{equation*}
$$

where $\left(\xi_{I}, \eta_{I}\right), I=1$ to 4 , are the nodal coordinates of the parent element shown in Fig. 4.4. They are given by

$$
\left[\xi_{i I}\right]=\left[\begin{array}{l}
\left.\xi_{I}\right\rceil  \tag{E4.2.3}\\
\eta_{I}
\end{array}\right]=\left[\begin{array}{cccc}
-1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1
\end{array}\right]
$$

Since (E4.2.1) also holds for $t=0$, we can write

$$
\left\{\begin{array}{c}
X(\xi)  \tag{E4.2.4}\\
Y(\xi)
\end{array}\right\}=\left\{\begin{array}{c}
X_{I} \\
Y_{I}
\end{array}\right\} N_{I}(\xi)
$$

where $X_{I}, Y_{I}$ are the coordinates in the undeformed configuration. The nodal velocities are given by

$$
\left\{\begin{array}{l}
v_{x}(\xi, t)  \tag{E4.2.5}\\
v_{y}(\xi, t)
\end{array}\right\}=\left\{\begin{array}{l}
v_{x I}(t) \\
v_{y I}(t)
\end{array}\right\} N_{I}(\xi)
$$

which is the material time derivative of the expression for the displacement.
Rate-of-Deformation and Internal Nodal Forces. The map (E4.2.1) is not invertible for the shape functions given by (E4.2.2). Therefore it is impossible to write explicit expressions for the element coordinates in terms of $x$ and $y$, and the derivatives of the shape functions are evaluated by using implicit differentiation. Referring to (4.4.47) we have

$$
N_{I, \mathbf{x}}^{T}=\left[\begin{array}{ll}
N_{I, x} & N_{I, y}
\end{array}\right]=N_{I, \xi}^{T} \mathbf{x}_{, \xi}^{-1}=\left[\begin{array}{ll}
N_{I, \xi} & N_{I, \eta}
\end{array}\right]\left[\begin{array}{cc}
\xi_{, x} & \xi_{, y}  \tag{E4.2.6}\\
\eta_{, x} & \eta_{, y}
\end{array}\right]
$$

The Jacobian of the current configuration with respect to the element coordinates is given by

For the 4-node quadrilateral the above is

$$
\mathbf{x}_{, \xi}=\sum_{I=1}^{4}\left[\begin{array}{ll}
x_{I}(t) \xi_{I}\left(1+\eta_{I} \eta\right) & x_{I}(t) \eta_{I}\left(1+\xi_{I} \xi\right)  \tag{E4.2.7b}\\
y_{I}(t) \xi_{I}\left(1+\eta_{I} \eta\right) & y_{I}(t) \eta_{I}\left(1+\xi_{I} \xi\right)
\end{array}\right]
$$

In the above, the summation has been indicated explicitly because the index $I$ appears three times. As can be seen from the RHS, the Jacobian matrix is a function of time. The inverse of $\mathbf{F}_{\xi}$ is given by

$$
\mathbf{x}_{, \xi}^{-1}=\frac{1}{J_{\xi}}\left[\begin{array}{cc}
y_{, \eta} & -x_{, \eta}  \tag{E4.2.7c}\\
-y_{, \xi} & x_{, \xi}
\end{array}\right], \quad J_{\xi}=x_{, \xi} y_{, \eta}-x_{, \eta} y_{, \xi}
$$

The gradients of the shape functions for the 4-node quadrilateral with respect to the element coordinates are given by

$$
\mathbf{N}_{, \xi}^{T}=\left[\partial N_{I} / \partial \xi_{i}\right]=\left[\begin{array}{ll}
\partial N_{1} / \partial \xi & \partial N_{1} / \partial \eta \\
\partial N_{2} / \partial \xi & \partial N_{2} / \partial \eta \\
\partial N_{3} / \partial \xi & \partial N_{3} / \partial \eta \\
\partial N_{4} / \partial \xi & \partial N_{4} / \partial \eta
\end{array}\right]=\frac{1}{4}\left[\begin{array}{ll}
\xi_{1}\left(1+\eta_{1} \eta\right) & \eta_{1}\left(1+\xi_{1} \xi\right) \\
\xi_{2}\left(1+\eta_{2} \eta\right) & \eta_{2}\left(1+\xi_{2} \xi\right) \\
\xi_{3}\left(1+\eta_{3} \eta\right) & \eta_{3}\left(1+\xi_{3} \xi\right) \\
\xi_{4}\left(1+\eta_{4} \eta\right) & \eta_{4}\left(1+\xi_{4} \xi\right)
\end{array}\right]
$$

The gradients of the shape functions with respect to the spatial coordinates can then be computed by

$$
\mathcal{B}_{I}=N_{I, \mathbf{x}}^{T}=N_{I, \xi}^{T} \mathbf{x}_{, \xi}^{-1}=\left[\begin{array}{cc}
\xi_{1}\left(1+\eta_{1} \eta\right) & \eta_{1}\left(1+\xi_{1} \xi\right)  \tag{E4.2.8a}\\
\xi_{2}\left(1+\eta_{2} \eta\right) & \eta_{2}\left(1+\xi_{2} \xi\right) \\
\xi_{3}\left(1+\eta_{3} \eta\right) & \eta_{3}\left(1+\xi_{3} \xi\right) \\
\xi_{4}\left(1+\eta_{4} \eta\right) & \eta_{4}\left(1+\xi_{3} \xi\right)
\end{array}\right] \frac{1}{J_{\xi}}\left[\begin{array}{cc}
y_{, \eta} & -y_{, \xi} \\
-x_{, \eta} & x_{, \xi}
\end{array}\right]
$$

and the velocity gradient is given by Eq. (4.5.3)

$$
\begin{equation*}
\mathbf{L}=\mathbf{v}_{I} \mathcal{B}_{I}^{T}=\mathbf{v}_{I} N_{I, \mathbf{x}}^{T} \tag{E4.2.8b}
\end{equation*}
$$

For a 4-node quadrilateral which is not rectangular, the velocity gradient, and hence the rate-ofdeformation, is a rational function because $J_{\xi}=\operatorname{det}\left(\mathbf{x}_{, \xi}\right)$ appears in the denominator of $\mathbf{x}_{, \xi}$ and hence in $\mathbf{L}$. The determinant $J_{\xi}$ is a linear function in $(\xi, \eta)$.

The nodal internal forces are obtained by (4.5.6), which gives

$$
\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\left[\begin{array}{ll}
f_{x I} & f_{y I}
\end{array}\right]^{i n t}=\int_{\Omega} \mathcal{B}_{I}^{T} \sigma d \Omega=\int_{\Omega}\left[\begin{array}{ll}
N_{I, x} & N_{I, y}
\end{array}\right]\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{x y}  \tag{E4.2.9}\\
\sigma_{x y} & \sigma_{y y}
\end{array}\right] d \Omega
$$

The integration is performed over the parent domain. For this purpose, we use

$$
\begin{equation*}
d \Omega=J_{\xi} a d \xi d \eta \tag{E4.2.10}
\end{equation*}
$$

where $a$ is the thickness. The internal forces are then given by (4.4.11), which when written out for two dimensions gives:

$$
\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\left[\begin{array}{ll}
f_{x I} & f_{y I}
\end{array}\right]^{i n t}=\int_{\Delta}\left[\begin{array}{ll}
N_{I, x} & N_{I, y}
\end{array}\right]\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{x y}  \tag{E4.2.11}\\
\sigma_{x y} & \sigma_{y y}
\end{array}\right] a J_{\xi} d \Delta
$$

where $N_{I, i}$ is given in Eq. (E4.2.8a). Equation (E4.2.14) applies to any isoparametric element in two dimensions. The integrand is a rational function of the element coordinates, since $J_{\xi}$ appears in the denominator (see Eq. (4.2.8a)), so analytic quadrature of the above is not feasible. Therefore numerical quadrature is generally used. For the 4 -node quadrilateral, 2x2 Gauss quadrature is full quadrature. However, for full quadrature, as discussed in Chapter 8, the element locks for incompressible and nearly incompressible materials in plane strain problems. Therefore, selective-reduced quadrature as described in Section 4.5.4, in which the volumetric stress is underintegrated, must be used for the four-node quadrilateral for plane strain problems when the material response is nearly incompressible, as in elastic-plastic materials.

The displacement for a 4-node quadrilateral is linear along each edge. Therefore, the external nodal forces are identical to those for the 3-node triangle, see Eqs. (E4.1.29-E4.1.33).

Mass Matrix. The consistent mass matrix is obtained by using (4.4.52), which gives

$$
\tilde{\mathbf{M}}=\int_{\Omega_{0}}\left[\begin{array}{l}
N_{1}  \tag{E4.2.12}\\
N_{2} \\
N_{3} \\
N_{4}
\end{array}\right]\left[\begin{array}{llll}
N_{1} & N_{2} & N_{3} & N_{4}
\end{array}\right] \rho_{0} d \Omega_{0}
$$

We use

$$
\begin{equation*}
d \Omega_{0}=J_{\xi}^{0}(\xi, \eta) a_{0} d \xi d \eta \tag{E4.2.13}
\end{equation*}
$$

where $J_{\xi}^{0}(\xi, \eta)$ is the determinant of the Jacobian of the transformation of the parent element to the initial configuration $a_{0}$ is the thickness of the undeformed element. The expression for $\tilde{\mathbf{M}}$ when evaluated in the parent domain is given by

$$
\tilde{\mathbf{M}}=\int_{-1}^{+1-1} \int^{\lceil+1}\left|\begin{array}{cccc}
N_{1}^{2} & N_{1} N_{2} & N_{1} N_{3} & N_{1} N_{4}  \tag{E4.2.14}\\
\text { symmetric } & N_{2}^{2} & N_{2} N_{3} & N_{2} N_{4}
\end{array}\right| \rho_{0} a_{0} J_{\xi}^{0}(\xi, \eta) d \xi d \eta
$$

The matrix is evaluated by numerical quadrature. This mass matrix can be expanded to an 8 x 8 matrix using the same procedure described for the triangle in the previous example.

A lumped, diagonal mass matrix can be obtained by using Lobatto quadrature with the quadrature points coincident with the nodes. If we denote the integrand of Eq. (E4.2.14) by $\mathbf{m}\left(\xi_{I}, \eta_{I}\right)$, then Lobatto quadrature gives

$$
\begin{equation*}
\overline{\mathbf{M}}=\sum_{I=1}^{4} \mathbf{m}\left(\xi_{I}, \eta_{I}\right) \tag{E4.2.15}
\end{equation*}
$$

Alternatively, the lumped mass matrix can be obtained by apportioning the total mass of the element equally among the four nodes. The total mass is $\rho_{0} A_{0} a_{0}$ when $a_{0}$ is constant, so dividing it among the four nodes gives

$$
\begin{equation*}
\overline{\mathbf{M}}=\frac{1}{4} \rho_{0} A_{0} a_{0} \mathbf{I}_{4} \tag{E4.2.16}
\end{equation*}
$$

where $\mathbf{I}_{4}$ is the unit matrix of order 4.
Example 4.3. Three Dimensional Isoparametric Element. Develop the expressions for the rate-of-deformation, the nodal forces and the mass matrix for three dimensional isoparametric elements. An example of this class of elements, the eight-node hexahedron, is shown in Fig. 4.5.


Fig. 4.5. Parent element and current configuration for an 8-node hexahedral element.
Motion and Strain Measures. The motion of the element is given by

$$
\left\{\begin{array}{l}
x  \tag{E4.3.1}\\
y \\
z
\end{array}\right\}=N_{I}(\xi)\left\{\begin{array}{l}
x_{I}(t) \\
y_{I}(t) \\
z_{I}(t)
\end{array}\right\} \quad \xi=(\xi, \eta, \zeta)
$$

where the shape functions for particular elements are given in Appendix C. Equation (E4.3.1) also holds at time $t=0$, so

$$
\left\{\begin{array}{c}
X  \tag{E4.3.2}\\
Y \\
Z
\end{array}\right\}=N_{I}(\xi)\left\{\begin{array}{l}
X_{I} \\
Y_{I} \\
Z_{I}
\end{array}\right\}
$$

The velocity field is given by

$$
\left\{\begin{array}{l}
v_{x}  \tag{E4.3.3}\\
v_{y} \\
v_{z}
\end{array}\right\}=N_{I}(\xi)\left\{\begin{array}{l}
v_{x I} \\
v_{y I} \\
v_{z I}
\end{array}\right\}
$$

The velocity gradient is obtained from Eq. (4.5.3), giving

$$
\begin{align*}
& \mathcal{B}_{I}^{T}=\left[\begin{array}{lll}
N_{I, x} & N_{I, y} & N_{I, z}
\end{array}\right]  \tag{E4.3.4}\\
& \mathbf{L}=\mathbf{v}_{I} \mathcal{B}_{I}^{T}=\left\{\begin{array}{l}
v_{x I} \\
v_{y I} \\
v_{z I}
\end{array}\right\}\left[\begin{array}{lll}
N_{I, x} & N_{I, y} & N_{I, z}
\end{array}\right]  \tag{E4.3.5}\\
&=\left[\begin{array}{lll}
v_{x I} N_{I, x} & v_{x I} N_{I, y} & v_{x I} N_{I, z} \\
v_{y I} N_{I, x} & v_{y I} N_{I, y} & v_{y I} N_{I, z} \\
v_{z I} N_{I, x} & v_{z I} N_{I, y} & v_{z I} N_{I, z}
\end{array}\right] \tag{E4.3.6}
\end{align*}
$$

The derivatives with respect to spatial coordinates are obtained in terms of derivatives with respect to the element coordinates by Eq. (4.4.37).

$$
\begin{align*}
& N_{I, \mathbf{x}}^{T}=N_{I, \xi}^{T} \mathbf{x}_{, \xi}^{-1}  \tag{E4.3.7}\\
& \mathbf{x}, \xi \equiv \mathbf{F}_{\xi} \mathbf{x}, \xi=\mathbf{x}_{I} N_{I, \xi}^{T}=\left\{\begin{array}{l}
x_{I} \\
y_{I} \\
z_{I}
\end{array}\right\}\left[\begin{array}{lll}
N_{I, \xi} & N_{I, \eta} & N_{I, \zeta}
\end{array}\right] \tag{E4.3.8}
\end{align*}
$$

The deformation gradient can be computed by Eqs. (3.2.10), (E4.3.1) and (E4.3.7):

$$
\begin{equation*}
\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{X}}=\mathbf{x}_{I} N_{I, \mathbf{X}}=\mathbf{x}_{I} N_{I, \xi}^{T} \mathbf{X}_{, \xi}^{-1} \equiv \mathbf{x}_{I} N_{I, \xi}^{T}\left(\mathbf{F}_{\xi}^{0}\right)^{-1} \tag{E4.3.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{X}_{, \xi} \equiv \mathbf{F}_{\xi}^{0}=\mathbf{X}_{I} N_{I, \xi}^{T} \tag{E4.3.10}
\end{equation*}
$$

The Green strain is then computed by Eq. (3.3.5); a more accurate procedure is described in Example 4.12.

Internal Nodal Forces. The internal nodal forces are obtained by Eq. (4.5.6):

$$
\left(\mathbf{f}_{I}^{\mathrm{int}}\right)^{T}=\left[f_{x I}, f_{y I}, f_{z I}\right]^{i n t}=\int_{\Omega} \mathcal{B}_{I}^{T} \sigma d \Omega=\int_{\Delta}\left[\begin{array}{lll}
N_{I, x} & N_{L, y} & N_{I, z}
\end{array}\right]\left[\begin{array}{lll}
\sigma_{x x} & \sigma_{x y} & \sigma_{x z}  \tag{E4.3.11}\\
\sigma_{x y} & \sigma_{y y} & \sigma_{y z} \\
\sigma_{x z} & \sigma_{y z} & \sigma_{z z}
\end{array} \|_{\xi} d \Delta\right.
$$

The integral is evaluated by numerical quadrature, using the quadrature formula (4.5.26).
External Nodal Forces. We consider first the nodal forces due to the body force. By Eq. (4.4.13), we have

$$
\begin{align*}
& f_{i I}^{e x t}=\int_{\Omega} N_{I} \rho b_{i} d \Omega=\int_{\Delta} N_{I}(\xi) \rho(\xi) b_{i}(\xi) J_{\xi} d \Delta  \tag{E4.3.12a}\\
& \left\{\begin{array}{l}
f_{x I} \\
f_{y I} \\
f_{z I}
\end{array}\right\}^{e x t}=\int_{-1-1-1}^{1} \int_{I}^{1} \int_{I}(\xi) \rho(\xi)\left\{\begin{array}{l}
b_{x}(\xi) \\
\left.b_{y}(\xi)\right\} \\
b_{z}(\xi)
\end{array}\right\} J_{\xi} d \xi d \eta d \zeta \tag{E4.3.12b}
\end{align*}
$$

where we have transformed the integral to the parent domain. The integral over the parent domain is evaluated by numerical quadrature.

To obtain the external nodal forces due to an applied pressure $\mathbf{t}=-p \mathbf{n}$, we consider a surface of the element. For example, consider the external surface corresponding with the parent element surface $\zeta=-1$; see Fig. 4.6. The nodal forces for any other surface are constructed similarly.

On any surface, any dependent variable can be expressed as a function of two parent coordinates, in this case they are $\xi$ and $\eta$. The vectors $\mathbf{x}, \xi$ and $\mathbf{x},{ }_{\eta}$ are tangent to the surface. The vector $\mathbf{x},{ }_{\xi} \times \mathbf{x}, \eta_{\eta}$ is in the direction of the normal $\mathbf{n}$ and as shown in any advanced calculus text, its magnitude is the surface Jacobian, so we can write

$$
\begin{equation*}
p \mathbf{n} d \Gamma=p\left(\mathbf{x}_{\xi} \times \mathbf{x}_{\boldsymbol{\eta}}\right) d \xi d \eta \tag{E4.3.13}
\end{equation*}
$$

For a pressure load, only the normal component of the traction is nonzero. The nodal external forces are then given by

$$
\begin{equation*}
f_{i I}^{e x t}=\int_{\Gamma} t_{i} N_{I} d \Gamma=-\int_{\Gamma} p n_{i} N_{I} d \Gamma=-\int_{-1-1}^{1} \int_{i j k}^{1} p e_{i j k} x_{j, \xi} x_{k, \eta} N_{I} d \xi d \eta \tag{E4.3.14}
\end{equation*}
$$

where we have used (E4.3.13) in indicial form in the last step. In matrix form the above is

$$
\begin{equation*}
\mathbf{f}_{I}^{e x t}=-\int_{\Gamma} p N_{I} \mathbf{x}_{, \xi} \times \mathbf{x}_{, \eta} d \Gamma \tag{E4.3.16}
\end{equation*}
$$

We have used the convention that the pressure is positive in compression. We can expand the above by using Eq. (4.4.1) to express the tangent vectors in terms of the shape functions and writing the cross product in determinant form, giving

$$
\mathbf{f}_{I}^{e x t}=f_{x l} \mathbf{e}_{x}+f_{y l} \mathbf{e}_{y}+f_{z l} \mathbf{e}_{z}=-\int_{-1}^{1} \int_{-1}^{1} p N_{I} \operatorname{det}\left[\begin{array}{ccc}
\mathbf{e}_{x} & \mathbf{e}_{y} & \mathbf{e}_{z}  \tag{E4.3.17}\\
x_{J} N_{J \xi} & y_{J} N_{J, \xi} & z_{J} N_{J, \xi} \\
x_{K} N_{K \eta} & y_{K} N_{K, \eta} & z_{K} N_{K, \eta}
\end{array}\right] d \xi d \eta
$$

This integral can readily be evaluated by numerical quadrature over the loaded surfaces of the parent element.

Example 4.4. Axisymmetric Quadrilateral. The expressions for the rate-of-deformation and the nodal forces for the axisymmetric quadrilateral element are developed. The element is shown in Fig. 4.7. The domain of the element is the volume swept out by rotating the quadrilateral $2 \pi$ radians about the axis of symmetry, the $z$-axis. The expressions in indicial notation, Eqs. (4.5.3) and (4.5.6), are not directly applicable since they do not apply to curvilinear coordinates.


Fig. 4.7. Current configuration of quadrilateral axisymmetric element; the element consists of the volume generated by rotating the quadrilateral $2 \pi$ radians about the z -axis.

In this case, the isoparametric map relates the cylindrical coordinates $[r, z]$ to the parent element coordinates $[\xi, \eta]$ :

$$
\left\{\begin{array}{c}
r(\xi, \eta, t)  \tag{E4.4.1}\\
z(\xi, \eta, t)
\end{array}\right\} r=\left\{\begin{array}{l}
r_{I}(t) \\
z_{I}(t)
\end{array}\right\} N_{I}(\xi, \eta)
$$

where the shape functions $N_{I}$ are given in (E4.2.20. The expression for the rate-of-deformation is based on standard expressions of the gradient in cylindrical coordinates (the expression are identical to the expressions for the linear strain):

$$
\left\{\begin{array}{c}
D_{r}  \tag{E4.4.2}\\
D_{z} \\
D_{\theta} \\
2 D_{r z}
\end{array}\right\}=\left[\begin{array}{cc}
\frac{\partial}{\partial r} & 0 \\
0 & \frac{\partial}{\partial z} \\
\frac{1}{r} & 0 \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial r}
\end{array}\right]\left\{\begin{array}{l}
v_{r} \\
v_{z}
\end{array}\right\}=\left\{\begin{array}{c}
\frac{\partial v_{r}}{\partial r} \\
\frac{\partial v_{z}}{\partial z} \\
\frac{v_{r}}{r} \\
\frac{\partial v_{r}}{\partial z}+\frac{\partial v_{z}}{\partial r}
\end{array}\right\}
$$

The conjugate stress is

$$
\begin{equation*}
\{\boldsymbol{\sigma}\}^{T}=\left[\sigma_{r}, \sigma_{z}, \sigma_{\theta}, \sigma_{r z}\right] \tag{E4.4.3}
\end{equation*}
$$

The velocity field is given by

$$
\begin{align*}
& \left\{\begin{array}{l}
v_{r} \\
v_{z}
\end{array}\right\}=N_{I}(\xi, \eta)\left\{\begin{array}{l}
v_{r I} \\
v_{z I}
\end{array}\right\}=\left[\begin{array}{cccccccc}
N_{1} & 0 & N_{2} & 0 & N_{3} & 0 & N_{4} & 0 \\
0 & N_{1} & 0 & N_{2} & 0 & N_{3} & 0 & N_{4}
\end{array}\right] \dot{\mathbf{d}}  \tag{E4.4.4}\\
& \dot{\mathbf{d}}^{T}=\left[v_{r 1}, v_{z 1}, v_{r 2}, v_{z 2}, v_{r 3}, v_{z 3}, v_{r 4}, v_{z 4}\right] \tag{E4.4.5}
\end{align*}
$$

The submatrices of the $\mathbf{B}$ matrix are given from Eq. (E4.4.2) by

$$
[B]_{I}=\left[\begin{array}{cc}
\frac{\partial N_{I}}{\partial r} & 0  \tag{E4.4.6}\\
0 & \frac{\partial N_{I}}{\partial z} \\
\frac{N_{I}}{r} & 0 \\
\frac{\partial N_{I}}{\partial z} & \frac{\partial N_{I}}{\partial r}
\end{array}\right]
$$

The derivatives in (E4.4.6) now have to be expressed in terms of derivates with respect to the parent element coordinates. Rather than obtaining these with a matrix product, we just write out the expressions using (E4.2.7c) with $x, y$ replaced by $r, z$, which gives

$$
\begin{align*}
& \frac{\partial N_{I}}{\partial r}=\frac{1}{J_{\xi}}\left(\frac{\partial z}{\partial \eta} \frac{\partial N_{I}}{\partial \xi}-\frac{\partial r}{\partial \eta} \frac{\partial N_{I}}{\partial \eta}\right)  \tag{E4.4.7a}\\
& \frac{\partial N_{I}}{\partial z}=\frac{1}{J_{\xi}}\left(\frac{\partial r}{\partial \xi} \frac{\partial N_{I}}{\partial \eta}-\frac{\partial z}{\partial \xi} \frac{\partial N_{I}}{\partial \xi}\right) \tag{E4.4.7b}
\end{align*}
$$

where

$$
\begin{array}{ll}
\frac{\partial z}{\partial \eta}=z_{I} \frac{\partial N_{I}}{\partial \eta} & \frac{\partial z}{\partial \xi}=z_{I} \frac{\partial N_{I}}{\partial \xi} \\
\frac{\partial r}{\partial \eta}=r_{I} \frac{\partial N_{I}}{\partial \eta} & \frac{\partial r}{\partial \xi}=r_{I} \frac{\partial N_{I}}{\partial \xi} \tag{E4.4.8b}
\end{array}
$$

The nodal forces are obtained from (4.5.14), which yields

$$
\begin{equation*}
\mathbf{f}_{I}^{i n t}=\int_{\Omega} \mathbf{B}_{I}^{T}\{\sigma\} d \Omega=2 \pi \int_{\Delta} \mathbf{B}_{I}^{T}\{\sigma\} J_{\xi} r d \Delta \tag{E4.4.9}
\end{equation*}
$$

where $\mathbf{B}_{I}$ is given by (E4.4.6) and we have used $d \Omega=2 \pi r J_{\xi} d \Delta$ where $r$ is given by Eq. (E4.4.1). The factor $2 \pi$ is often omitted from all nodal forces, i.e. the element is taken to be the volume generated by sweeping the quadrilateral by one radian about the z-axis in Fig. 4.7.

Example 4.5. Master-Slave Tieline. A master slave tieline is shown in Figure 4.5. Tielines are frequently used to connect parts of the mesh which use different element sizes, for they are more convenient than connecting the elements of different sizes by triangles or tetrahedra. Continuity of the motion across the tieline is enforced by constraining the motion of the slave nodes to the linear field of the adjacent edge connecting the master nodes. In the following, the resulting nodal forces and mass matrix are developed by the transformation rules of Section 4.5.5.


- master nodes

O slave nodes

Fig. 4.8. Exploded view of a tieline; when joined together, the velocites of nodes 3 and 5 equal the nodal velocities of nodes 1 and 2 and the velocity of node 4 is given in terms of nodes 1 and 2 by a linear constraint.

The slave node velocities are given by the kinematic constraint that the velocities along the two sides of the tieline must remain compatible, i.e. $C^{0}$. This constraint can be expressed as a linear relation in the nodal velocities, so the relation corresponding to Eq. (4.5.35) can be written as

$$
\left\{\begin{array}{c}
\hat{\mathbf{v}}_{M}  \tag{E4.5.1}\\
\hat{\mathbf{v}}_{S}
\end{array}\right\}=\left[\begin{array}{l}
\mathbf{I} \\
\mathbf{A}
\end{array}\right]\left\{\mathbf{v}_{M}\right\} \quad \text { so } \quad \mathbf{T}=\left[\begin{array}{c}
\mathbf{I} \\
\mathbf{A}
\end{array}\right]
$$

where the matrix $\mathbf{A}$ is obtained from the linear constraint and the superposed hats indicate the velocities of the disjoint model before the two sides are tied together. We denote the nodal forces of the disjoint model at the slave nodes and master nodes by $\hat{\mathbf{f}}_{S}$ and $\hat{\mathbf{f}}_{M}$, respectively. Thus, $\hat{\mathbf{f}}_{S}$ is the matrix of nodal forces assembled from the elements on the slave side of the tieline and $\hat{\mathbf{f}}_{M}$ is the matrix of nodal forces assembled from the elements on the master side of the tieline. The nodal forces for the joined model are then given by Eq. (4.5.36):

$$
\left\{\mathbf{f}_{M}\right\}=\mathbf{T}^{T}\left\{\begin{array}{l}
\hat{\mathbf{f}}_{M}  \tag{E4.5.2}\\
\hat{\mathbf{f}}_{S}
\end{array}\right\}=\left[\begin{array}{ll}
\mathbf{I} & \mathbf{A}^{T}
\end{array}\right]\left[\begin{array}{c}
\hat{\mathbf{f}}_{M} \\
\hat{\mathbf{f}}_{S}
\end{array}\right\}
$$

where $\mathbf{T}$ is given by (E4.5.1). As can be seen from the above, the master nodal forces are the sum of the master nodal forces for the disjoint model and the transformed slave node forces. These formulas apply to both the external and internal nodal forces.

The consistent mass matrix is given by Eq. (4.5.39):

$$
\mathbf{M}=\mathbf{T}^{T} \mathbf{M T}=\left[\begin{array}{ll}
\mathbf{I} & \mathbf{A}^{T}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{M}_{M} & 0  \tag{E4.5.3}\\
0 & \mathbf{M}_{s}
\end{array}\right]\left[\begin{array}{l}
\mathbf{I} \\
\mathbf{A}
\end{array}\right]=\mathbf{M}_{M}+\mathbf{A}^{T} \mathbf{M}_{s} \mathbf{A}
$$

We illustrate these transformations in more detail for the 5 nodes which are numbered in Fig. 4.8. The elements are 4-node quadrilaterals, so the velocity along any edge is linear. Slave nodes 3 and 5 are coincident with master nodes 1 and 2, and slave node 4 is at a distance $\xi \ell$ from node 1 , where $\ell=\left\|\mathbf{x}_{2}-\mathbf{x}_{1}\right\|$. Therefore,

$$
\begin{equation*}
\mathbf{v}_{3}=\mathbf{v}_{1}, \quad \mathbf{v}_{5}=\mathbf{v}_{2}, \quad \mathbf{v}_{4}=\xi \mathbf{v}_{2}+(1-\xi) \mathbf{v}_{1} \tag{E4.5.4}
\end{equation*}
$$

and Eq. (E4.5.1) can be written as

$$
\left\{\begin{array}{l}
\mathbf{v}_{1}  \tag{E4.5.5}\\
\mathbf{v}_{2} \\
\mathbf{v}_{3} \\
\mathbf{v}_{4} \\
\mathbf{v}_{5}
\end{array}\right\}=\left[\begin{array}{cc}
\mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} \\
\mathbf{I} & 0 \\
(1-\xi) \mathbf{I} & \xi \mathbf{I} \\
0 & \mathbf{I}
\end{array}\right\}\left\{\begin{array}{cc}
\mathbf{I} & \mathbf{0} \\
\mathbf{\mathbf { v } _ { 1 }} \\
\mathbf{v}_{2}
\end{array}\right\} \quad \mathbf{T}=\left[\begin{array}{cc}
\mathbf{0} & \mathbf{I} \\
\mathbf{I} & 0 \\
(1-\xi) \mathbf{I} & \xi \mathbf{I} \\
0 & \mathbf{I}
\end{array}\right]
$$

The nodal forces are then given by

$$
\left[\begin{array}{l}
\mathbf{f}_{1}  \tag{E4.5.6}\\
\mathbf{f}_{2}
\end{array}\right]=\left[\begin{array}{ccccc}
\mathbf{I} & \mathbf{0} & \mathbf{I} & (1-\xi) \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \xi \mathbf{I} & \mathbf{I}
\end{array}\right]\left[\begin{array}{c}
\hat{\mathbf{f}}_{1} \\
\hat{\mathbf{f}}_{2} \\
\hat{\mathbf{f}}_{3} \\
\hat{\mathbf{f}}_{4} \\
\hat{\mathbf{f}}_{5}
\end{array}\right\}
$$

The force for master node 1 is

$$
\begin{equation*}
\mathbf{f}_{1}=\hat{\mathbf{f}}_{1}+\hat{\mathbf{f}}_{3}+(1-\xi) \hat{\mathbf{f}}_{5} \tag{E4.5.7}
\end{equation*}
$$

Both components of the nodal force transform identically; the transformation applies to both internal and external nodal forces. The mass matrix is transformed by Eq. (4.5.39) using T as given in Eq. (E4.5.1).

If the two lines are only tied in the normal direction, a local coordinate system needs to be set up at the nodes to write the linear constraint. The normal components of the nodal forces are then related by a relation similar to Eq. (4.5.7), whereas the tangential components remain independent.

### 4.6 COROTATIONAL FORMULATIONS

In structural elements such as bars, beams and shells, it is awkward to deal with fixed coordinate systems. Consider for example a rotating rod such as shown in Fig. 3.6. Initially, the only nonzero stress is $\sigma_{x}$, whereas $\sigma_{y}$ vanishes. Subsequently, as the rod rotates it is awkward to express the state of uniaxial stress in a simple way in terms of the global components of the stress tensor.

A natural approach to overcoming this difficulty is to embed a coordinate system in the bar and rotate the embedded system with the rod. Such coordinate systems are known as corotational coordinates. For example, consider a coordinate system, $\hat{\mathbf{x}}=[\hat{x}, \hat{y}]$ for a rod so that $\hat{x}$ always connects nodes 1 and 2, as shown in Fig. 4.9. A uniaxial state of stress can then always be described by the condition that $\hat{\sigma}_{y}=\hat{\sigma}_{x y}=0$ and that $\hat{\sigma}_{x}$ is nonzero. Similarly the rate-ofdeformation of the rod is described by the component $\hat{D}_{x}$.

There are two approaches to corotational finite element formulations:

1. a coordinate system is embedded at each quadrature point and rotated with material in some sense.
2. a coordinate system is embedded in an element and rotated with the element.

The first procedure is valid for arbitrarily large strains and large rotations. A major consideration in corotational formulations lies in defining the rotation of the material. The polar decomposition theorem can be used to define a rotation which is independent of the coordinate system. However, when particular directions of the material have a large stiffness which must be represented accurately, the rotation provided by a polar decomposition does not necessarily provide the best rotation for a Cartesian coordinate system; this is illustrated in Chapter 5.

A remarkable aspect of corotational theories is that although the corotational coordinate is defined only at discrete points and is Cartesian at these points, the resulting finite element
formulation accurately reproduces the behavior of shells and other complex structures. Thus, by using a corotational formulation in conjunction with a "degenerated continuum" approximation, the complexities of curvilinear coordinate formulations of shells can be avoided. This is further discussed in Chapter 9, since this is particularly attractive for the nonlinear analysis of shells.

For some elements, such as a rod or the constant strain triangle, the rigid body rotation is the same throughout the element. It is then sufficient to embed a single coordinate system in the element. For higher order elements, if the strains are small, the coordinate system can be embedded so that it does not rotate exactly with the material as described later. For example, the corotational coordinate system can be defined to be coincident to one side of the element. If the rotations relative to the embedded coordinate system are of order $\theta$, then the error in the strains is of order $\theta^{2}$. Therefore, as long as $\theta^{2}$ is small compared to the strains, a single embedded coordinate system is adequate. These applications are often known as small-strain, large rotation problems; see Wempner (1969) and Belytschko and Hsieh(1972).

The components of a vector $\mathbf{v}$ in the corotational system are related to the global components by

$$
\begin{equation*}
\hat{v}_{i}=R_{j i} v_{j} \quad \text { or } \quad \hat{\mathbf{v}}=\mathbf{R}^{\mathrm{T}} \mathbf{v} \quad \text { and } \quad \mathbf{v}=\mathbf{R} \hat{\mathbf{v}} \tag{4.6.1}
\end{equation*}
$$

where $\mathbf{R}$ is an orthogonal transformation matrix defined in Eqs. (3.2.24-25) and the superposed " $\wedge$ " indicates the corotational components.

The corotational components of the finite element approximation to the velocity field can be written as

$$
\begin{equation*}
\hat{v}_{i}(\xi, t)=N_{I}(\xi) \hat{v}_{i I}(t) \tag{4.6.2}
\end{equation*}
$$

This expression is identical to (4.4.32) except that it pertains to the corotational components. Equation (4.6.2) can be obtained from (4.4.32) by multiplying both sides by $\mathbf{R}^{T}$.

The corotational components of the velocity gradient tensor are given by

$$
\begin{equation*}
\hat{L}_{i j}=\frac{\partial \hat{v}_{i}}{\partial \hat{x}_{j}}=\frac{\partial N_{I}(\xi)}{\partial \hat{x}_{j}} \hat{v}_{i I}(t)=\hat{\mathcal{B}}_{j I} \hat{v}_{i I} \quad \text { or } \quad \hat{\mathbf{L}}=\hat{\mathbf{v}}_{I} \frac{\partial N_{I}}{\partial \hat{\mathbf{x}}}=\hat{\mathbf{v}}_{I} N_{I, \hat{\mathbf{x}}}^{T}=\hat{\mathbf{v}}_{I} \hat{\mathcal{B}}_{I}^{T} \tag{4.6.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\mathcal{B}}_{j I}=\frac{\partial N_{I}}{\partial \hat{x}_{j}} \tag{4.6.4}
\end{equation*}
$$

The corotational rate-of-deformation tensor is then given by

$$
\begin{equation*}
\hat{D}_{i j}=\frac{1}{2}\left(\hat{L}_{i j}+\hat{L}_{j i}\right)=\frac{1}{2}\left(\frac{\partial \hat{v}_{i}}{\partial \hat{x}_{j}}+\frac{\partial \hat{v}_{j}}{\partial \hat{x}_{i}}\right) \tag{4.6.5}
\end{equation*}
$$

The corotational formulation is used only for the evaluation of internal nodal forces. The external nodal forces and the mass matrix are sually evaluated in the global system as before. The
the semi-discrete equations of motion are treated in terms of global components. We therefore concern ourselves only with the evaluation of the internal nodal forces in the corotational formulation.

The expression for $\hat{\mathbf{f}}_{I}^{\text {int }}$ in terms of corotational components is developed as follows. We start with the standard expression for the nodal internal forces, Eq. (4.5.5):

$$
\begin{equation*}
f_{i I}^{i n t}=\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega \quad \text { or } \quad\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\int_{\Omega} N_{I, \mathbf{x}}^{T} \sigma d \Omega \tag{4.6.6}
\end{equation*}
$$

By the chain rule and Eq. (4.6.1)

$$
\begin{equation*}
\frac{\partial N_{I}}{\partial x_{j}}=\frac{\partial N_{I}}{\partial \hat{x}_{k}} \frac{\partial \hat{x}_{k}}{\partial x_{j}}=\frac{\partial N_{I}}{\partial \hat{x}_{k}} R_{j k} \quad \text { or } \quad N_{I, \mathbf{x}}=\mathbf{R} N_{I, \hat{\mathbf{x}}} \tag{4.6.7}
\end{equation*}
$$

Substituting the transformation for the Cauchy stress into the corotational stress, Box 3.2, and Eq. (4.6.7) into Eq. (4.6.6), we obtain

$$
\begin{equation*}
\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\int_{\Omega} N_{I, \hat{\mathbf{x}}}^{T} \mathbf{R}^{T} \mathbf{R} \hat{\sigma} \mathbf{R}^{T} d \Omega \tag{4.6.8}
\end{equation*}
$$

and using the orthogonality of $\mathbf{R}$, we have

$$
\begin{equation*}
\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\int_{\Omega} N_{I, \hat{\mathbf{x}}}^{T} \hat{\sigma} \mathbf{R}^{T} d \Omega \quad \text { or } \quad\left[f_{i l}^{i n t}\right]^{T}=f_{I i}^{i n t}=\int_{\Omega} \frac{\partial N_{I}}{\partial \hat{x}_{j}} \hat{\sigma}_{j k} R_{k i}^{T} d \Omega \tag{4.6.9}
\end{equation*}
$$

Comparing the above to the standard expression for the nodal internal forces, (4.6.5), we can see that the expressions are similar, but the stress is expressed in the corotational system and the rotation matrix $\mathbf{R}$ now appears. In the expression on the right, the indices on $\mathbf{f}^{i n t}$ have been exchanged so that the expression can be converted to matrix form.

If we use the $\hat{\mathcal{B}}$ matrix defined by Eq. (4.6.4) we can write

$$
\begin{equation*}
\left(\mathbf{f}_{I}^{\text {int }}\right)^{T}=\int_{\Omega} \hat{\mathcal{B}}_{I}^{T} \hat{\sigma} \mathbf{R}^{T} d \Omega \quad \quad \mathbf{f}_{\text {int }}^{T}=\int \mathcal{B}^{T} \hat{\sigma} \mathbf{R}^{T} d \Omega \tag{4.6.10}
\end{equation*}
$$

Corresponding relations for the internal nodal forces can be developed in Voigt notation:

$$
\begin{equation*}
\mathbf{f}_{I}^{\text {int }}=\int_{\Omega} \mathbf{R}^{T} \hat{\mathbf{B}}_{I}^{T}\{\hat{\sigma}\} d \Omega \quad \text { where } \quad\{\hat{\mathbf{D}}\}=\hat{\mathbf{B}}_{I} \hat{\mathbf{v}}_{I} \tag{4.6.11}
\end{equation*}
$$

and $\hat{\mathbf{B}}_{I}$ is obtained from $\hat{B}_{I}$ by the Voigt rule.
The rate of the corotational Cauchy stress is objective (frame-invariant), so the constitutive equation can be expressed directly as a relationship between the rate of the corotational Cauchy stress and the corotational rate-of-deformation

$$
\begin{equation*}
\frac{D \hat{\sigma}}{D t}=S^{\hat{\sigma} \hat{D}}(\hat{\mathbf{D}}, \hat{\sigma}, e t c) \tag{4.6.12}
\end{equation*}
$$

In particular, for hypoelastic material,

$$
\begin{equation*}
\frac{D \hat{\sigma}}{D t}=\hat{\mathbf{C}}: \hat{\mathbf{D}} \quad \text { or } \quad \frac{D \hat{\sigma}_{i j}}{D t}=\hat{C}_{i j k l} \hat{D}_{k l} \tag{4.6.13}
\end{equation*}
$$

where the elastic response matrix is also expressed in terms of the corotational components. An attractive feature of the above relation is that the $\hat{\mathbf{C}}$ matrix for anisotropic materials need not be changed to reflect rotations. Since the coordinate system rotates with the material, material rotation has no effect on $\hat{\mathbf{C}}$. On the other hand, for an anisotropic material, the $\mathbf{C}$ matrix changes as the material rotates.

Example 4.6. Rods in Two Dimensions. A two-node element is shown in Fig. 4.9. The element uses linear displacement and velocity fields. The corotational coordinate $\hat{x}$ is chosen to coincide with the axis of the element at all times as shown. Obtain an expression for the corotational rate-of-deformation and the internal nodal forces. Then the methodology is extended to a three-node rod.


Fig. 4.9. Two-node rod element showing initial configuration and current configuration and the corotational coordinate.

The displacement and velocity fields are linear in $\hat{x}$ and given by

$$
\begin{align*}
& \left\{\begin{array}{l}
x \\
y
\end{array}\right\}=\left\{\begin{array}{ll}
x_{1} & x_{2} \\
y_{1} & y_{2}
\end{array}\right\}\left\{\begin{array}{c}
1-\xi \\
\xi
\end{array}\right\} \\
& \left\{\begin{array}{l}
\hat{v}_{x} \\
\hat{v}_{y}
\end{array}\right\}=\left[\begin{array}{ll}
\hat{v}_{x 1} & \hat{v}_{x 2} \\
\hat{v}_{y 1} & \hat{v}_{y 2}
\end{array}\right]\left\{\begin{array}{c}
1-\xi \\
\xi
\end{array}\right\}  \tag{E4.6.1}\\
& \xi=\frac{\hat{x}}{\ell}
\end{align*}
$$

where $\ell$ is the current length of the element. The corotational velocities are related to the global components by the vector transformation Eq. (E4.6.1):

$$
\left\{\begin{array}{l}
v_{x I}  \tag{E4.6.2}\\
v_{y I}
\end{array}\right\}=\mathbf{R}\left\{\begin{array}{l}
\hat{v}_{x I} \\
\hat{v}_{y I}
\end{array}\right\}, \mathbf{R}=\left[\begin{array}{ll}
R_{x \hat{x}} & R_{x \hat{y}} \\
R_{y \hat{x}} & R_{y \hat{y}}
\end{array}\right]=\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right]=\frac{1}{\ell}\left[\begin{array}{cc}
x_{21} & -y_{21} \\
y_{21} & x_{21}
\end{array}\right]
$$

A state of uniaxial stress is assumed; the only nonzero stress is $\hat{\sigma}_{x}$ which is the stress along the axis of the bar element. Since $\hat{x}$ rotates with the bar element, $\hat{\sigma}_{x}$ is the axial stress for any orientation of the element. Only the axial component of the rate-of-deformation tensor, $\hat{D}_{x}$, contributes to the internal power. It is given the derivative of the velocity field (E4.6.1):

$$
\hat{D}_{x}=\frac{\partial \hat{v}_{x}}{\partial \hat{x}}=\left[N_{I, \hat{x}}\right]\left\{\begin{array}{l}
\hat{v}_{x 1}  \tag{E4.6.3}\\
\hat{v}_{x 2}
\end{array}\right\}=\frac{1}{\ell}\left[\begin{array}{ll}
-1 & +1
\end{array}\right]\left\{\begin{array}{l}
\hat{v}_{x 1} \\
\hat{v}_{x 2}
\end{array}\right\}=\hat{\mathbf{B}} \hat{\mathbf{v}} \quad \hat{\mathbf{B}}=\left[\begin{array}{l}
N_{I, \hat{x}}
\end{array}\right]=\frac{1}{\ell}\left[\begin{array}{ll}
-1 & +1
\end{array}\right]
$$

Nodal Internal Forces. The nodal internal forces are obtained from Eq. (4.6.8), which can be rewritten as

$$
\begin{equation*}
\left[\mathbf{f}_{I i}\right]^{i n t}=\int_{\Omega} \frac{\partial N_{I}}{\partial \hat{x}_{j}} \hat{\sigma}_{j k} R_{k i}^{T} d \Omega=\int_{\Omega} \frac{\partial N_{I}}{\partial \hat{x}} \hat{\sigma}_{x x} R_{\hat{x} \hat{x}}^{T} d \Omega=\int_{\Omega} \hat{\mathbf{B}}^{T} \hat{\sigma}_{x x} R_{\hat{x} i}^{T} d \Omega \tag{E4.6.4}
\end{equation*}
$$

where the second expression omits the many zeros which appear in the more general expression; the subscripts on the internal nodal forces have been interchanged. Substituting (E4.6.2) and (E4.6.3) into the above gives

$$
\left[f_{I i}\right]^{\text {int }}=\int \frac{1}{\ell}\left[\begin{array}{l}
-1  \tag{E4.6.6}\\
{[1}
\end{array}\right]\left[\hat{\sigma}_{x}\right]\left[\begin{array}{ll}
\cos \theta & \sin \theta] d \Omega \\
\hline
\end{array}\right.
$$

If we assume the stress is constant in the element, we can evaluate the integral by multiplying the integral by the volume of the element, $V=A \ell$, which gives

$$
\left[f_{I i}\right]^{i n t}=\left[\begin{array}{ll}
f_{1 x} & f_{1 y}  \tag{E4.6.7}\\
f_{2 x} & f_{2 y}
\end{array}\right]=A \hat{\sigma}_{x}\left[\begin{array}{cc}
-\cos \theta & -\sin \theta \\
\cos \theta & \sin \theta
\end{array}\right]
$$

The above result shows that the nodal forces are along the axis of the rod and equal and opposite at the two nodes.

The stress-strain law in this element is computed in the corotational system. Thus, the rate form of the hypoelastic law is

$$
\begin{equation*}
\frac{D \hat{\sigma}_{x}}{D t}=E \hat{D}_{x} \tag{E4.6.8}
\end{equation*}
$$

where $E$ is a tangent modulus in uniaxial stress. The rotation terms which appear in the objective rates are not needed, since the coordinate system is corotational.

To evaluate the nodal forces, the current cross-sectional area $A$ must be known. The change in area can then be expressed in terms of the transverse strains; the exact formula depends on the shape of the cross-section. For a rectangular cross-section

$$
\begin{equation*}
\dot{A}=A\left(\hat{D}_{y}+\hat{D}_{z}\right) \tag{E4.6.9a}
\end{equation*}
$$

Computation of internal nodal forces from one-dimensional rod. The internal nodal forces can also be obtained by computing the corotational components as in Example 2.8.1, Eq. (E2.2.8) and then transforming by Eq. (4.5.40). In the corotational system, the nodal forces are given by Eq. (E2.8.8), so we write this equation in the corotational system:

$$
\hat{\mathbf{f}}^{i n t}=\left\{\begin{array}{l}
\hat{f}_{x 1}  \tag{E4.6.10}\\
\hat{f}_{x 2}
\end{array}\right\}^{i n t}=\int_{\Omega} \frac{1}{l}\left[\begin{array}{l}
-1 \\
+1
\end{array}\right] \hat{\sigma}_{x} A d x
$$

Since the we are considering a slender rod with no stiffness normal to its axis, the transverse snodal forces vanish, i.e. $\hat{f}_{y 1}=\hat{f}_{y 2}=0$.

Voigt notation. In Voigt procedures, the element equations are usually developed by starting with the equations in the local, corotational cooordinates. The global components of the nodal forces can then be obtained by the transformation equations, (4.5.40). We first define $\mathbf{T}$ by relating the local degrees-of-freedom (which are conjugate to $\hat{\mathbf{f}}^{\hat{i n t}}$ ) to the four degrees-of-freedom of the element:

$$
\left\{\begin{array}{l}
\hat{v}_{x 1}  \tag{E4.6.11}\\
\hat{v}_{x 2}
\end{array}\right\}=\left[\begin{array}{cccc}
\cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta
\end{array}\right]\left\{\begin{array}{l}
v_{x 1} \\
v_{y 1} \\
v_{x 2} \\
v_{y 2}
\end{array}\right\} \text { so } \mathbf{T}=\left[\begin{array}{cccc}
\cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta
\end{array}\right]
$$

which defines the $\mathbf{T}$ matrix. Using Eq. (4.5.36), $\mathbf{f}=\mathbf{T}^{T} \mathbf{f}$, and assuming the stress is constant in the element then gives

$$
\mathbf{f}^{i n t}=\left\{\begin{array}{l}
f_{x 1}  \tag{E4.6.12}\\
f_{y 1} \\
f_{x 2} \\
f_{y 2}
\end{array}\right\}^{\text {int }}=\mathbf{T}^{T} \mathbf{f}^{i n t}=\left[\begin{array}{cc}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & \cos \theta \\
0 & \sin \theta
\end{array}\right] A \hat{\sigma}_{x}\left\{\begin{array}{c}
-1 \\
1
\end{array}\right\}=A \hat{\sigma}_{x}\left\{\begin{array}{c}
-\cos \theta \\
-\sin \theta \\
\cos \theta \\
\sin \theta
\end{array}\right\}
$$

which is identical to (E4.6.7).
Three-Node Element. We consider the three-node curved rod element shown in Fig. 4.10. The configurations, displacement, and velocity are given by quadratic fields. The expression for the nodal internal forces will be developed by the corotational approach.


Fig. 4.10. Initial, current, and parent elements for a three-node rod; the corotational base vector $\hat{\mathbf{e}}_{x}$ is tangent to the current configuration.

The initial and current configurations are given by

$$
\begin{equation*}
\mathbf{X}(\xi, t)=\mathbf{X}_{I}(t) N_{I}(\xi) \quad \mathbf{x}(\xi, t)=\mathbf{x}_{I}(t) N_{I}(\xi) \tag{E4.6.13}
\end{equation*}
$$

where

$$
\left[N_{I}\right]=\left[\begin{array}{lll}
\frac{1}{2} \xi(\xi-1) & 1-\xi^{2} & \frac{1}{2} \xi(\xi+1) \tag{E4.6.14}
\end{array}\right]
$$

The displacement and velocity are given by

$$
\begin{equation*}
\mathbf{u}(\xi, t)=\mathbf{u}_{I}(t) N_{I}(\xi) \quad \mathbf{v}(\xi, t)=\mathbf{v}_{I}(t) N_{I}(\xi) \tag{E4.6.15}
\end{equation*}
$$

The corotational system is defined at each point of the rod (in practice it is needed only at the quadrature points). Let $\hat{\mathbf{e}}_{x}$ be tangent to the rod, so

$$
\begin{equation*}
\hat{e}_{x}=\frac{\mathbf{x}_{, \xi}}{\|\mathbf{X}, \xi\|} \text { where } \mathbf{x}_{, \xi}=\mathbf{x}_{I} N_{I, \xi}(\xi) \tag{E4.6.16}
\end{equation*}
$$

The normal to the element is given by

$$
\begin{equation*}
\hat{\mathbf{e}}_{y}=\mathbf{e}_{z} \times \hat{\mathbf{e}}_{x} \text { where } \mathbf{e}_{z}=[0,0,1] \tag{E4.6.17}
\end{equation*}
$$

The rate of deformation is given by

$$
\hat{D}_{x}=\frac{\partial \hat{\hat{x}}_{x}}{\partial \hat{x}}=\frac{\partial \hat{v}_{x}}{\partial \xi} \frac{\partial \xi}{\partial \hat{x}}=\frac{1}{\| \mathbf{x}^{\prime}, \xi} \frac{\partial \hat{v}_{x}}{\partial \xi} \text { must be explained-may be wrong }(\mathrm{E} 4.6 .18)
$$

From Eq. (E4.6.15) and Eq. (E4.6.18)

$$
\begin{equation*}
\hat{v}_{x}=N_{I}(\xi)\left(R_{x x} v_{x I}+R_{y x} v_{y I}\right) \tag{E4.6.19}
\end{equation*}
$$

the rate-of-deformation is given by

$$
\hat{D}_{x}=\frac{1}{\left\|\mathbf{x}_{t \xi}\right\|} N_{I \xi}(\xi)\left\{\begin{array}{l}
v_{x l}  \tag{E4.6.20}\\
v_{y l}
\end{array}\right\}
$$

The above shows the $\hat{\mathcal{B}}_{I}$ matrix to be

$$
\begin{equation*}
\hat{\mathcal{B}}_{I}=\frac{1}{\mid \mathbf{x}_{, \xi} \|} N_{I \xi} \tag{E4.6.21}
\end{equation*}
$$

The nodal internal forces are then given by

$$
\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\left[\begin{array}{ll}
f_{x I} & f_{y I}
\end{array}\right]^{j n t}=\int_{-1}^{1} A \hat{\mathcal{B}}_{I} \hat{\sigma}_{x}\left\|\mathbf{x}_{5}\right\|\left[\begin{array}{ll}
R_{x \hat{x}} & R_{y x} \tag{E4.6.22}
\end{array}\right] d \xi
$$

An interesting feature of the above development is that it avoids curvilinear tensors completely. However, the rate-of-deformation as computed here is correct; Exercize ?? shows how Eq. (E4.6.20) reproduces the correct result for a curved bar.

Example 4.7. Triangular Element. Develop the expression for the velocity strain and the nodal internal forces for a three-node triangle using the corotational approach.


initial configuration $\Omega_{0}^{e}$
current configuration $\Omega$

Fig. 4.11 Triangular three-node element treated by corotational coordinate system.
The element in its initial and current configurations is shown in Fig. 4.11. The corotational system is initially at an angle of $\theta_{0}$ with the global coordinate system; in the following, $\theta_{0}$ is often chosen to vanish, but for an anistropic material it may be desirable to orient the initial $\hat{x}$-axis in a direction of anisotropy, for example, in a composite material it may be useful to orient $\hat{x}$ in a fiber direction. The current angle of the corotational coordinate system is $\theta$. We discuss how to compute this angle subsequently.

The motion can be expressed in terms of the triangular coordinates, as in Example 4.1.

$$
\left\{\begin{array}{l}
x  \tag{E4.7.1}\\
y
\end{array}\right\}=\left[\begin{array}{lll}
x_{1} & x_{2} & x_{3} \\
y_{1} & y_{2} & y_{3}
\end{array}\right]\left\{\begin{array}{l}
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right\}
$$

The displacement and velocity fields in the element are then given by

$$
\begin{align*}
& \left\{\begin{array}{l}
\hat{u}_{x} \\
\hat{u}_{y}
\end{array}\right\}=\left[\begin{array}{ccc}
\hat{u}_{x 1} & \hat{u}_{x 2} & \hat{u}_{x 3} \\
\hat{u}_{y 1} & \hat{u}_{y 2} & \hat{u}_{y 3}
\end{array}\right]\left\{\begin{array}{l}
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right\}  \tag{E4.7.2}\\
& \left\{\begin{array}{l}
\hat{v}_{x} \\
\hat{v}_{y}
\end{array}\right\}=\left[\begin{array}{lll}
\hat{v}_{x 1} & \hat{v}_{x 2} & \hat{v}_{x 3} \\
\hat{v}_{y 1} & \hat{v}_{y 2} & \hat{v}_{y 3}
\end{array}\right]\left\{\begin{array}{l}
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right\} \tag{E4.7.3}
\end{align*}
$$

The derivatives of the shape functions with respect to the corotational coordinate system are given by the counterpart of (E4.1.5):

$$
\left[\partial N_{I} / \partial \hat{x}_{j}\right] \equiv\left[\partial \xi_{I} / \partial \hat{x}_{j}\right]=\frac{1}{2 A}\left[\begin{array}{ll}
\hat{y}_{23} & \hat{x}_{32}  \tag{E4.7.4}\\
\hat{y}_{31} & \hat{x}_{13} \\
\hat{y}_{12} & \hat{x}_{21}
\end{array}\right] \equiv \hat{\mathcal{B}}
$$

The corotational components of the rate-of-deformation are given by

$$
\begin{equation*}
\hat{\mathbf{D}}=\frac{1}{2}\left(\hat{\mathcal{B}}_{I} \mathbf{v}_{I}^{T}+\mathbf{v}_{I} \hat{\mathcal{B}}_{I}^{T}\right) \tag{E4.7.5}
\end{equation*}
$$

The nodal internal forces are given by Eq. (4.6.10):

$$
\begin{equation*}
\left[f_{I i}\right]^{i n t}=\int_{\Omega} \hat{\mathcal{B}}_{l j} \hat{\sigma}_{j k} R_{k i}^{T} d \Omega=\int_{\Omega} \frac{\partial \xi_{I}}{\partial \hat{x}_{j}} \hat{\sigma}_{j k} R_{k i}^{T} d \Omega \tag{E4.7.6}
\end{equation*}
$$

Writing out the matrices using (E4.6.2) and (E4.7.4) gives

$$
\left[\begin{array}{ll}
f_{1 x} & f_{1 y}  \tag{E4.7.7}\\
f_{2 x} & f_{2 y} \\
f_{3 x} & f_{3 y}
\end{array}\right]^{\text {int }}=\int_{A} \frac{1}{2 A}\left[\begin{array}{ll}
\hat{y}_{23} & \hat{x}_{32} \\
\hat{y}_{31} & \hat{x}_{13} \\
\hat{y}_{12} & \hat{x}_{21}
\end{array}\right]\left[\begin{array}{cc}
\hat{\sigma}_{x} & \hat{\sigma}_{x y} \\
\hat{\sigma}_{x y} & \hat{\sigma}_{y}
\end{array}\right]\left[\begin{array}{cc}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{array}\right] a d A
$$

The rotation of the coordinate system can be obtained in several ways:

1. by polar decomposition;
2. by rotating the corotational coordinate system with a material line in the element, e.g. a preferred direction in a composite;
3. by rotating the corotational coordinate system with a side of the element (this is only correct for small strain problems).
To use polar decomposition, the same approach as described in Section 3, Example ?? is used.

### 4.7. TOTAL LAGRANGIAN FORMULATION

4.7.1. Governing Equations. The physical principles which govern the total Lagrangian formulation are the same as those for the updated Lagrangian formulation, which were given in Section 4.2. The form of the governing equations is different, but as has been seen in Chapter 3, they express the same physical principles and can be obtained by transforming the associated conservation equations from Eulerian to Lagrangian form.

Similarly, the finite element equations for the total Lagrangian formulation can be obtained by transforming the equations for the updated Lagrangian formulation. It is only necessary to transform the integrals to the reference (undeformed) domain and transform the stress and strain measures to the Lagrangian type. This approach is used in Section 4.7.2, and for most readers Section 4.7.2 and the following examples will suffice as an introduction to the total Lagrangian formulation. However, for readers who would like to see the entire structure of the total Lagrangian formulation or prefer to learn it first, Section 4.8 gives a development of the weak form in the total Lagrangian description, followed by the direct derivation of the finite element equations from this weak form.

The governing equations are given in both tensor form and indicial form in Box 4.5. We have chosen to use the nominal stress $\mathbf{P}$ in the momentum equation, because the resulting momentum equation and its weak form are simpler than for the PK2 stress. However, the nominal stress is awkward in constitutive equations because of its lack of symmetry, so we have used the PK2 stress for constitutive equations. Once the PK2 stress has been evaluated by the constitutive
equations, the nominal stress stress can then easily be obtained by a transformation given in Box 3.2, Eq. (B4.5.5). The constitutive equation can relate the Cauchy stress $\sigma$ to the rate-ofdeformation $\mathbf{D}$. The stress would then be converted to the nominal stress $\mathbf{P}$ prior to evaluation of the nodal forces. However, this entails additional transformations and hence additional computational expense, so when the constitutive equations are expressed in terms of $\sigma$ it is advantageous to use the updated Lagrangian formulation.

## Box 4.5

## Governing Equations For Total Lagrangian Formulation

conservation of mass

$$
\begin{equation*}
\rho J=\rho_{0} J_{0}=\rho_{0} \tag{B4.5.1}
\end{equation*}
$$

conservation of linear momentum

$$
\begin{equation*}
\nabla_{X} \cdot \mathbf{P}+\rho_{0} \mathbf{b}=\rho_{0} \ddot{\mathbf{u}} \quad \text { or } \quad \frac{\partial P_{j i}}{\partial X_{j}}+\rho_{0} b_{i}=\rho_{0} \ddot{u}_{i} \tag{B4.5.2}
\end{equation*}
$$

conservation of angular momentum:

$$
\begin{equation*}
\mathbf{F} \cdot \mathbf{P}=\mathbf{P}^{T} \cdot \mathbf{F}^{T} \quad \text { or } \quad F_{i j} P_{j k}=F_{k j} P_{j i} \tag{B4.5.3}
\end{equation*}
$$

conservation of energy

$$
\begin{align*}
& \rho_{0} \dot{w}^{i n t}=\dot{\mathbf{F}}^{T}: \mathbf{P}-\nabla_{X} \cdot \overline{\mathbf{q}}+\rho_{0} s \quad \text { or } \quad \rho_{0} \dot{w}^{i n t}=\dot{F}_{i j} P_{j i}-\frac{\partial \bar{q}_{i}}{\partial X_{i}}+\rho_{0} s  \tag{B4.5.4}\\
& \text { where } \overline{\mathbf{q}}=J \mathbf{F}^{-1} \mathbf{q}
\end{align*}
$$

constitutive equation

$$
\begin{equation*}
\mathbf{S}=\boldsymbol{S}(\mathbf{E}, . . e t c) \quad \mathbf{P}=\mathbf{S} \cdot \mathbf{F}^{T} \tag{B4.5.5}
\end{equation*}
$$

measure of strain

$$
\begin{equation*}
\mathbf{E}=\frac{1}{2}\left(\mathbf{F}^{T} \cdot \mathbf{F}-\mathbf{I}\right) \quad \text { or } \quad E_{i j}=\frac{1}{2}\left(F_{k i} F_{k j}-\delta_{i j}\right) \tag{B4.5.6}
\end{equation*}
$$

boundary conditions

$$
\begin{align*}
& n_{j}^{0} P_{j i}=\tilde{t}_{i}^{0} \quad \text { or } \quad \mathbf{e}_{i} \cdot \mathbf{n}^{0} \cdot \mathbf{P}=\mathbf{e}_{i} \cdot \overline{\mathbf{t}}_{0} \quad \text { on } \Gamma_{t_{i}}^{0}, \quad u_{i}=\bar{u}_{i} \text { on } \Gamma_{u_{i}}^{0}  \tag{B4.5.7}\\
& \Gamma_{t_{i}}^{0} \cup \Gamma_{u_{i}}^{0}=\Gamma^{0} \quad \Gamma_{t_{i}}^{0} \cap \Gamma_{u_{i}}^{0}=0 \quad \text { for } \mathrm{i}=1 \text { to } n_{s} \tag{B4.5.8}
\end{align*}
$$

initial conditions

$$
\begin{align*}
& \mathbf{P}(\mathbf{X}, 0)=\mathbf{P}_{0}(\mathbf{X}) \text { in }  \tag{B4.5.9}\\
& \mathbf{u}(\mathbf{X}, 0)=\mathbf{u}_{0}(\mathbf{X}) \tag{B4.5.10}
\end{align*}
$$

internal continuity conditions

$$
\begin{equation*}
\left\langle n_{j}^{0} P_{j i}\right\rangle=0 \text { on } \Gamma_{i n t}^{0} \tag{B4.5.11}
\end{equation*}
$$

The nominal stress is conjugate to the material time derivative of the deformation tensor, $\dot{\mathbf{F}}$, see Box 3.4. Thus in (B4.5.4) the internal work is expressed in terms of these two tensors. Note that we have used the left divergence of $\mathbf{P}$ (see (B.5.1.2)), so $\mathbf{n}^{0}$ appears before $\mathbf{P}$ in the traction expression; if the order is reversed the resulting matrix corresponds to the transpose of $\mathbf{P}$, which is the PK1 stress; see Section 3.4.1. The PK1 stress is also frequently used, so it is important to note the distinction between these two stress tensors. The traction is obtained in terms of the nominal stress by putting the initial normal to the left, and the left divergence operator is used in the momentum equation. For the PK1 stress, the normal appears to the right and the right divergence is used in the momentum equation.

The deformation tensor $\mathbf{F}$ is not suitable as a measure of strain in constitutive equations since it does not vanish in rigid body rotation. Therefore constitutive equations in total Lagrangian formulations are usually formulated in terms of the Green strain tensor $\mathbf{E}$, which can be obtained from $\mathbf{F}$. In the continuum mechanics literature, one often sees constitutive equations expressed as $\mathbf{P}=\mathbf{P}(\mathbf{F})$, which gives the impression that the constitutive equation uses $\mathbf{F}$ as a measure of strain.
In fact, when writing $\mathbf{P}(\mathbf{F})$, it is implicit that the constitutive stress depends on $\mathbf{F}^{T} \mathbf{F}$ (i.e., $\mathbf{E}+\mathbf{I}$, where the unit matrix I makes no difference) or some other measure of deformation which is independent of rigid body rotation. Similarly, the nominal stress $\mathbf{P}$ in constitutive equations is incorporated so it satisfies conservation of angular momentum, Eq. (B4.5.3).

As in any mechanical system, the same component of traction and displacement cannot be prescribed at any point of a boundary, but one of these must be prescribed; see Eqs. (B4.5.7B4.5.8). In the Lagrangian formulation, tractions are prescribed in units of force per undeformed area.

The total Lagrangian formulation can be obtained in two ways:

1. transforming the finite element equations for the updated Lagrangian fomulation to the initial (reference) configuration and expressing it in terms of Lagrangian variables.
2. by developing the weak form in terms of the initial configuration and Lagrangian variables and then using this weak form to obtain discrete equations.
We will begin with the first approach since it is quicker and more convenient. The second approach is only recommended for intensive courses or for those who prefer the total Lagrangian formulation.
4.7.2. Total Lagrangian Finite Element Equations by Transformation. To obtain the discrete finite element equations for total Lagrangian formulation, we will transform each of the
nodal force expressions in the updated Lagrangian formulation, beginning with the internal nodal forces. The mass conservation equation (B4.5.1), $\rho J=\rho_{0}$, and the relation

$$
\begin{equation*}
d \Omega=J d \Omega_{0} \tag{4.7.1}
\end{equation*}
$$

will also be used. The internal nodal forces are given in the updated Lagrangian formulation by Eq. (4.4.10)

$$
\begin{equation*}
f_{i I}^{i n t}=\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \sigma_{j i} d \Omega \tag{4.7.2}
\end{equation*}
$$

Using the transformation from Box 3.2, $J \sigma_{j i}=F_{j k} P_{k i}=\frac{\partial x_{j}}{\partial X_{k}} P_{k i}$, we convert (4.7.2) to:

$$
\begin{equation*}
f_{i I}^{i n t}=\int_{\Omega} \frac{\partial N_{I}}{\partial x_{j}} \frac{\partial x_{j}}{\partial X_{k}} P_{k i} J^{-1} d \Omega \tag{4.7.3}
\end{equation*}
$$

Recognizing that the product of the first two terms is a chain rule expression of $\partial N_{I} / \partial X_{k}$ and using Eq. (4.7.1), we get

$$
\begin{equation*}
f_{i l}^{i n t}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{k}} P_{k i} d \Omega_{0}=\int_{\Omega_{0}} \mathcal{B}_{01 k} P_{k i} d \Omega_{0} \tag{4.7.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{B}_{0 k l}=\frac{\partial N_{I}}{\partial X_{k}} \tag{4.7.5}
\end{equation*}
$$

In matrix form, the above can be written as

$$
\begin{equation*}
\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\int_{\Omega_{0}} \mathcal{B}_{0}^{T} \mathbf{P} d \Omega_{0} \tag{4.7.6}
\end{equation*}
$$

The expression has been written in the above form to stress the analogy to the updated Lagrangian form: if $\mathcal{B}$ is replaced by $\mathcal{B}_{0}, \Omega$ by $\Omega_{0}$, and $\sigma$ by $\mathbf{P}$, we obtain the updated Lagrangian form from the above.

The external nodal forces are next obtained by transforming the updated Lagrangian expression to the total Lagrangian form. We start with Eq. (4.4.13)

$$
\begin{equation*}
f_{i I}^{e x t}=\int_{\Omega} N_{l} \rho b_{i} d \Omega+\int_{\Gamma_{t_{i}}} N_{l} \bar{t}_{i} d \Gamma \tag{4.7.7}
\end{equation*}
$$

Substituting Eq. (3.6.1), $\mathbf{\rho} \mathbf{b} d \Omega=\rho_{0} \mathbf{b} d \Omega_{0}$, and Eq. (3.4.4), $\overline{\mathbf{t}} d \Gamma=\mathbf{t}_{0} d \Gamma_{0}$, into Eq. (4.7.7) gives

$$
\begin{equation*}
f_{i l}^{e x t}=\int_{\Omega_{0}} N_{I} \rho_{0} b_{i} d \Omega_{0}+\int_{\Gamma_{i}^{0}} N_{I} \tilde{t}_{i}^{0} d \Gamma_{0} \tag{4.7.8}
\end{equation*}
$$

which is the total Lagrangian form of the external nodal forces. The two integrals are over the intial (reference) domain and boundary; note that $\rho_{0} \mathbf{b}$ is the body force per unit of the reference volume, see (3.6.1). This can be written in matrix form as:

$$
\begin{equation*}
\mathbf{f}_{I}^{e x t}=\int_{\Omega_{0}} N_{I} \rho_{0} \mathbf{b} d \Omega_{0}+\int_{\Gamma_{\Gamma_{i}}^{0}} N_{I} \mathbf{e}_{i} \cdot \overline{\mathbf{t}}_{0} d \Gamma \tag{4.7.9}
\end{equation*}
$$

The inertial nodal forces and the mass matrix were expressed in terms of the initial configuration in the development of the updated Lagrangian form, Eq. (4.4.50). Thus, all of the nodal forces can be expressed in terms of Lagrangian variables on the initial (reference) configuration by the transformations. The equations of motion for the total Lagrangian discretization are identical to that of the updated Lagrangian discretization, Eq. (4.4.48).

### 4.8 TOTAL LAGRANGIAN WEAK FORM

In this Section, we develop the weak form from the strong form in a total Lagrangian format. Subsequently, we will show that the weak form implies the strong form. The strong form consists of the momentum equation, Eq. (B4.5.2), the traction boundary condition, Eq. (B4.5.7), and the interior continuity conditions, Eq. (B4.5.11). We define the spaces for the test and trial functions as in Section 4.3:

$$
\begin{equation*}
\delta \mathbf{u}(\mathbf{X}) \in \mathcal{U} \mathcal{l}_{0}, \mathbf{u}(\mathbf{X}, t) \in \mathcal{U} \tag{4.8.1}
\end{equation*}
$$

where $\mathcal{U}$ is the space of kinematically admissible displacements and $\mathcal{U}_{0}$ is the same space with the additional requirement that the displacements vanish on displacement boundaries.

Strong Form to Weak Form. To develop the weak form, we multiply the momentum equation (B4.5.3) by the test function and integrate over the initial (reference) configuration:

$$
\begin{equation*}
\int_{\Omega_{0}} \delta u_{i}\left(\frac{\partial P_{j i}}{\partial X_{j}}+\rho_{0} b_{i}-\rho_{0} \ddot{u}_{i}\right) d \Omega_{0}=0 \tag{4.8.2}
\end{equation*}
$$

In the above, the nominal stress is a function of the trial displacements via the consitutive equation and the strain-displacement equation. This weak form is not useful because it requires the trial displacements to be $C^{1}$, since a derivative of the nominal stress appears in (4.8.2); see Sections 4.3.1-2.

To eliminate the derivative of the nominal stress from Eq. (4.8.2), the derivative product formula is used:

$$
\begin{equation*}
\int_{\Omega_{0}} \delta u_{i} \frac{\partial P_{j i}}{\partial X_{j}} d \Omega_{0}=\int_{\Omega_{0}} \frac{\partial}{\partial X_{j}}\left(\delta u_{i} P_{j i}\right) d \Omega_{0}-\int_{\Omega_{0}} \frac{\partial\left(\delta u_{i}\right)}{\partial X_{j}} P_{j i} d \Omega_{0} \tag{4.8.3}
\end{equation*}
$$

The first term of the RHS of the above can be expressed as a boundary integral by Gauss's theorem (3.5.6):

$$
\begin{equation*}
\int_{\Omega_{0}} \frac{\partial}{\partial X_{j}}\left(\delta u_{i} P_{j i}\right) d \Omega_{0}=-\int_{\Gamma_{0}} \delta u_{i} n_{j}^{0} P_{j i} d \Gamma_{0}+\int_{\Gamma_{i \boldsymbol{i} t}^{0}} \delta u_{i}\left\langle n_{j}^{0} P_{j i}\right\rangle d \Gamma_{0} \tag{4.8.4}
\end{equation*}
$$

From the strong form Eq. (B4.5.11), the last term vanishes. The first term on the RHS can be reduced to the traction boundary since $\delta u_{i}=0$ on $\Gamma_{u_{i}}^{0}$ and $\Gamma_{t_{i}}^{0}=\Gamma_{0}-\Gamma_{u_{i}}^{0}$, so

$$
\begin{equation*}
\int_{\Omega_{0}} \frac{\partial}{\partial X_{j}}\left(\delta u_{i} P_{j i}\right) d \Omega_{0}=\int_{\Gamma_{0}} \delta u_{i} n_{j}^{0} P_{j i} d \Gamma_{0}=\sum_{i=1}^{n_{s D}} \int_{\Gamma_{t i}^{0}} \delta u_{i} \tilde{t}_{i}^{0} d \Gamma_{0} \tag{4.8.5}
\end{equation*}
$$

where the last equality follows from the strong form. From Eq. (3.2.14)we note that

$$
\begin{equation*}
\delta F_{i j}=\delta\left(\frac{\partial u_{i}}{\partial X_{j}}\right) \tag{4.8.6}
\end{equation*}
$$

Substituting Eq. (4.8.5) into (4.8.3) and the result into (4.8.2) gives, after a change of sign and using (4.8.6):

$$
\begin{equation*}
\int_{\Omega_{0}}\left(\delta F_{i j} P_{j i}-\delta u_{i} \rho_{0} b_{i}+\delta u_{i} \rho_{0} \ddot{u}_{i}\right) d \Omega_{0}-\sum_{i=1}^{n_{s D}} \int_{\Gamma_{i i}^{0}} \delta u_{i} \dot{t}_{i}^{0} d \Gamma_{0}=0 \tag{4.8.7}
\end{equation*}
$$

or

$$
\begin{equation*}
\int_{\Omega_{0}}\left(\delta \mathbf{F}^{T}: \mathbf{P}-\rho_{0} \delta \mathbf{u} \cdot \mathbf{b}+\rho_{0} \delta \mathbf{u} \cdot \ddot{\mathbf{u}}\right) d \Omega_{0}-\sum_{i=1}^{n_{S D}} \int_{\Gamma_{i i}^{0}}\left(\delta \mathbf{u} \cdot \mathbf{e}_{i}\right)\left(\mathbf{e}_{i} \cdot \overline{\mathbf{t}}_{i}^{0}\right) d \Gamma_{0}=0 \tag{4.8.8}
\end{equation*}
$$

The above is the weak form of the momentum equation, traction boundary conditions, and interior continuity conditions. It is called the principle of virtual work, since each of the terms in Eq. (4.8.7) is a virtual work increment. The weak form is summarized in Box 4.6, in which physical names are ascribed to each of the terms.

This weak form can also be developed by replacing the test velocity by a test displacement in Eq. (4.3.9) and transforming each term to the reference configuration. The total Lagrangian weak form, Eq. (4.8.8), is thus simply a transformation of the updated Lagrangian weak form.

## Box 4.6 <br> Weak Form for Total Lagrangian Formulation: Principle of Virtual Work

WEAK FORM: if $u \in \mathcal{U}$ and

$$
\begin{equation*}
\delta \mathcal{W}^{\text {int }}(\delta \mathbf{u}, \mathbf{u})-\delta \mathcal{W}^{\text {ext }}(\delta \mathbf{u}, \mathbf{u})+\delta \mathcal{W}^{\text {inert }}(\delta \mathbf{u}, \mathbf{u})=0 \quad \forall \delta \mathbf{u} \in \mathcal{U}_{0} \tag{B4.6.1}
\end{equation*}
$$

then equilibrium, the traction boundary conditions and internal continuity conditions are satisfied.
In the above

$$
\begin{align*}
& \delta \mathcal{W}^{\text {int }}=\int_{\Omega_{0}} \delta \mathbf{F}^{T}: \mathbf{P} d \Omega_{0}=\int_{\Omega_{0}} \delta F_{i j} P_{j i} d \Omega_{0}  \tag{B4.6.2}\\
& \delta \mathcal{W}^{\text {ext }}=\int_{\Omega_{0}} \rho_{0} \delta \mathbf{u} \cdot \mathbf{b} d \Omega_{0}+\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t i}^{0}}\left(\delta \mathbf{u} \cdot \mathbf{e}_{i}\right)\left(\mathbf{e}_{i} \cdot \overline{\mathbf{t}}_{i}^{0}\right) d \Gamma_{0}=0 \\
&  \tag{B4.6.3}\\
& =\int_{\Omega_{0}} \delta u_{i} \rho_{0} b_{i} d \Omega_{0}+\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t i}^{0}} \delta u_{i} \tilde{t}_{i}^{0} d \Gamma_{0}  \tag{B4.6.4}\\
& \delta \mathcal{W}^{\text {inert }}=\int_{\Omega_{0}} \delta \mathbf{u} \cdot \rho_{0} \ddot{\mathbf{u}} d \Omega_{0}=\int_{\Omega_{0}} \delta u_{i} \rho_{0} \ddot{u}_{i} d \Omega_{0}
\end{align*}
$$

Weak Form to Strong Form. Next we deduce the strong form from the weak form. To avoid writing summations, we shall assume that on any part of the boundary, all traction or displacement components are prescribed; the reader can easily generalize the proof to mixed boundary conditions where for each component, either the displacement or traction component is prescribed. Substituting (B4.8.6) into the first term of (B4.8.7) and the derivative product rule gives

$$
\begin{equation*}
\int_{\Omega_{0}} \frac{\partial\left(\delta u_{i}\right)}{\partial X_{j}} P_{j i} d \Omega_{0}=\int_{\Omega_{0}}\left[\frac{\partial}{\partial X_{j}}\left(\delta u_{i} P_{j i}\right)-\delta u_{i} \frac{\partial P_{j i}}{\partial X_{j}}\right] d \Omega_{0} \tag{4.8.9}
\end{equation*}
$$

Gauss's theorem on the first term on the RHS then yields:

$$
\int_{\Omega_{0}} \frac{\partial\left(\delta u_{i}\right)}{\partial X_{j}} P_{j i} d \Omega_{0}=\sum_{i=1}^{n_{S D}} \int_{\Gamma_{t i}^{0}} \delta u_{i} n_{j}^{0} P_{j i} d \Gamma_{0}+\int_{\Gamma_{i n t}^{0}} \delta u_{i}\left\langle n_{j}^{0} P_{j i}\right\rangle d \Gamma_{0}-\int_{\Omega_{0}} \delta u_{i} \frac{\partial P_{j i}}{\partial X_{j}} d \Omega_{0}
$$

where the surface integral is changed to the traction boundary because $\delta u_{i}=0$ on $\Gamma_{u_{i}}^{0}$ and $\Gamma_{t_{i}}^{0}=\Gamma^{0}-\Gamma_{u_{i}}^{0}$.

Substituting (4.8.9) into (4.8.7) and collecting terms gives

$$
\begin{equation*}
0=\int_{\Omega_{0}} \delta u_{i}\left(\frac{\partial P_{j i}}{\partial X_{j}}+\rho_{0} b_{i}-\rho_{0} \ddot{u}_{i}\right) d \Omega_{0}+\sum_{i=1}^{n_{S D}} \int_{\Gamma_{i}^{0}} \delta u_{i}\left(n_{j}^{0} P_{j i}-\dot{t}_{i}^{0}\right) d \Gamma_{0}+\int_{\Gamma_{\text {int }}^{0}} \delta u_{i} \backslash n_{j}^{0} P_{j i} \backslash d \Gamma_{0}=0(4 \tag{4.8.10}
\end{equation*}
$$

Since the above holds for all $\delta \mathbf{u} \in \mathcal{U}_{0}$, it follows by the density theorem given in Section 4.3.2 that the momentum equation ( B 4.5 .2 ) holds on $\Omega_{0}$, the traction boundary conditions ( B 4.5 .8 ) hold on $\Gamma_{t}^{0}$, and the interior continuity conditions (B4.5.11) hold on $\Gamma_{i n t}^{0}$. Thus the weak form implies the momentum equation, the traction boundary conditions, and the interior continuity conditions.

### 4.9 FINITE ELEMENT SEMIDISCRETIZATION

4.9.1. Discrete Equations. We consider a Lagrangian mesh with the same properties as described in Section 4.4.1. The finite element approximation to the motion is given by

$$
\begin{equation*}
x_{i}(\mathbf{X}, t)=x_{i I}(t) N_{I}(\mathbf{X}) \tag{4.9.5}
\end{equation*}
$$

where $N_{I}(\mathbf{X})$ are the shape functions; as in the updated Lagrangian formulation, the shpae functions are functions of the material (Lagrangian) coordinates. The trial displacement field is given by

$$
\begin{equation*}
u_{i}(\mathbf{X}, t)=u_{i I}(t) N_{I}(\mathbf{X}) \quad \text { or } \quad \mathbf{u}(X, t)=\mathbf{u}_{I}(t) N_{I}(\mathbf{X}) \tag{4.9.1}
\end{equation*}
$$

The test functions, or variations, are not functions of time, so

$$
\begin{equation*}
\delta u_{i}(\mathbf{X})=\delta u_{i I} N_{I}(\mathbf{X}) \quad \text { or } \quad \delta \mathbf{u}(\mathbf{X})=\delta \mathbf{u}_{I} N_{I}(\mathbf{X}) \tag{4.9.2}
\end{equation*}
$$

As before, we will use indicial notation where all repeated indices are summed; upper case indices pertain to nodes and are summed over all relevant nodes and lower case indices pertain to components and are summed over the number of dimensions.

Taking material time derivatives of (4.9.1) gives the velocity and acceleration

$$
\begin{gather*}
\dot{u}_{i}(\mathbf{X}, t)=\dot{u}_{i I}(t) N_{I}(\mathbf{X})  \tag{4.9.3}\\
\ddot{u}_{i}(\mathbf{X}, t)=\ddot{u}_{i I}(t) N_{I}(\mathbf{X}) \tag{4.9.4}
\end{gather*}
$$

The deformation gradient is then given by

$$
\begin{equation*}
F_{i j}=\frac{\partial x_{i}}{\partial X_{j}}=\frac{\partial N_{I}}{\partial X_{j}} x_{i I} \tag{4.9.6}
\end{equation*}
$$

It is sometimes convenient to write the above as

$$
\begin{equation*}
F_{i j}=\mathcal{B}_{j I}^{0} u_{i I} \quad \text { where } \quad \mathcal{B}_{j I}^{0}=\frac{\partial N_{I}}{\partial X_{j}} \quad \text { so } \quad \mathbf{F}=\mathbf{x} \mathcal{B}_{0}^{T} \tag{4.9.7}
\end{equation*}
$$

$$
\begin{equation*}
\delta F_{i j}=\frac{\partial N_{I}}{\partial X_{j}} \delta x_{i I}=\frac{\partial N_{I}}{\partial X_{j}} \delta u_{i I} \quad \text { so } \quad \delta \mathbf{F}=\delta \mathbf{u} \mathcal{B}_{0}^{T} \tag{4.9.8}
\end{equation*}
$$

where we have used $\delta x_{i I}=\delta\left(X_{i I}+u_{i I}\right)=\delta u_{i I}$. Nodal forces will now be developed for each of the virtual weak terms.

Internal nodal forces. The internal nodal forces are defined in terms of the internal virtual work using

$$
\begin{equation*}
\delta \mathcal{W}^{i n t}=\delta u_{i I} f_{i I}^{i n t}=\int_{\Omega_{0}} \delta F_{i j} P_{j i} d \Omega_{0}=\delta u_{i I} \int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}} P_{j i} d \Omega_{0} \tag{4.9.9}
\end{equation*}
$$

where Eq. (4.9.7) has been used in the last step. Then the arbitrariness of $\delta u_{i I}$ yields

$$
\begin{equation*}
f_{i I}^{i n t}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}} P_{j i} d \Omega_{0} \quad \text { or } \quad f_{i I}^{i n t}=\int_{\Omega_{0}} \mathcal{B}_{j I}^{0} P_{j i} d \Omega_{0} \quad \text { or } \quad f_{i I}^{i n t}=\int_{\Omega_{0}} \mathcal{B}_{j i}^{0} P_{j i} d \Omega_{0} \tag{4.9.10}
\end{equation*}
$$

which is identical to (4.7.6), the expression developed by transformation.
External Nodal Forces. The external nodal forces are defined by equating the virtual external work (B4.6.3) to the virtual work of the external nodal forces:

$$
\begin{equation*}
\delta \mathcal{W}^{e x t}=\delta u_{i I} f_{i I}^{e x t}=\int_{\Omega_{0}} \delta u_{i} \rho_{0} b_{i} d \Omega_{0}+\int_{\Gamma_{t i}^{0}} \delta u_{i} \bar{t}_{i}^{0} d \Gamma_{0}=\delta u_{i I}\left\{\int_{\Omega_{0}} N_{I} \rho_{0} b_{i} d \Omega_{0}+\int_{\Gamma_{t i}^{0}} N_{I} \tilde{t}_{i}^{0} d \Gamma_{0}\right\} \tag{4.9.12a}
\end{equation*}
$$

This gives

$$
\begin{equation*}
f_{i I}^{e x t}=\int_{\Omega_{0}} N_{I} \rho_{0} b_{i} d \Omega_{0}+\int_{\Gamma_{t}^{0}} N_{I} \dot{t}_{i}^{0} d \Gamma_{0} \tag{4.9.12b}
\end{equation*}
$$

Mass Matrix: Using the inertial force (B4.6.4) and defining an equivalent nodal force gives

$$
\begin{equation*}
\delta \mathcal{M}=\delta u_{i I} f_{i I}^{i n e r t}=\int_{\Omega_{0}} \delta u_{i} \rho_{0} \ddot{u}_{i} d \Omega_{0} \tag{4.9.13}
\end{equation*}
$$

Substituting Eq. (4.9.2) and (4.9.4) in the right hand side of the above gives

$$
\begin{equation*}
\delta u_{i I} f_{i I}^{i n e r t}=\delta u_{i I} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0} \ddot{u}_{j J}=\delta u_{i I} M_{i j I J} \ddot{u}_{j J} \tag{4.9.14}
\end{equation*}
$$

Since the above holds for arbitrary $\delta \mathbf{u}$ and $\ddot{\mathbf{u}}$, it follows that the mass matrix is given by

$$
\begin{equation*}
M_{i j I J}=\delta_{i j} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0} \tag{4.9.15}
\end{equation*}
$$

Comparing this mass matrix $\mathbf{M}$ to that used for the updated Lagrangian formulation, Eq. (4.4.51), we see that they are identical, which is expected since we transformed the mass to the reference configuration to highlight its time invariance for a Lagrangian mesh.

Substituting the above expressions into the weak form, (B4.6.1), we have

$$
\begin{equation*}
\delta u_{i l}\left(f_{i I}^{i n t}-f_{i I}^{e x t}+M_{i j J} \ddot{u}_{j J}\right)=0 \quad \forall I, i \notin \Gamma_{u_{i}} \tag{4.9.16}
\end{equation*}
$$

Since the above for arbitrary values of all nodal displacement components that are not constrained by displacement boundary conditions, it follows that

$$
\begin{equation*}
M_{i j J J} \ddot{u}_{j J}+f_{i I}^{\text {int }}=f_{i I}^{e x t} \quad \forall I, i \notin \Gamma_{u_{i}} \tag{4.9.17}
\end{equation*}
$$

The above equations are identical to the governing equations for the updated Lagrangian formulation, as given in Box 5.5. The nodal forces in the updated and total Lagrangian formulations are expressed in terms of different variables and integrated over different domains, but from a fundamental viewpoint the updated Lagrangian formulation and the total Lagrangian formulation are identical. The numerical values for the nodal forces obtained by either formulation are also identical. Each of these formulations can be advantageous for certain constitutive equations or loadings by reducing the number of transformations which are needed.
4.9.2. Implementation. The procedure for the computation of the internal nodal forces is given in Box 4.7. In the procedure shown, the nodal forces are evaluated by numerical quadrature.

## Box 4.7

## Discrete Equations and Internal Force Computation in Total Lagrangian Formulation

Equations of Motion (discrete momentum equation)

$$
\begin{equation*}
M_{i j I J} \dot{v}_{j J}+f_{i I}^{\text {int }}=f_{i l}^{e x t} \text { for }(I, i) \notin \Gamma_{v_{i}} \tag{B4.8.1}
\end{equation*}
$$

Internal Nodal Forces

$$
\begin{aligned}
& f_{i I}^{\text {int }}=\int_{\Omega_{0}} \mathcal{B}_{0 I j} P_{j i} d \Omega_{0}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}} P_{j t} d \Omega_{0} \text { or }\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\int_{\Omega_{0}} \mathcal{B}_{0 I}^{T} \mathbf{P} d \Omega_{0} \\
& \mathbf{f}_{I}^{\text {int }}=\int_{\Omega_{0}} \mathbf{B}_{0 I}^{T}\{\mathbf{S}\} d \Omega_{0} \text { in Voigt notation }
\end{aligned}
$$

External Nodal Forces

$$
\begin{equation*}
f_{i I}^{e x t}=\int_{\Omega_{0}} N_{I} \rho_{0} b_{i} d \Omega_{0}+\int_{\Gamma_{t i}^{0}} N_{I} \bar{t}_{i}^{0} d \Gamma_{0} \text { or } \mathbf{f}_{I}^{e x t}=\int_{\Omega_{0}} N_{I} \rho_{0} \mathbf{b} d \Omega_{0}+\int_{\Gamma_{t i}^{0}} N_{I} \mathbf{e}_{i} \cdot \overline{\mathbf{t}}^{0} d \Gamma_{0} \tag{B4.8.3}
\end{equation*}
$$

Mass Matrix (total Lagrangian)

$$
\begin{align*}
& M_{i j I J}=\delta_{i j} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0}=\delta_{i j} \int_{\Delta} \rho_{0} N_{I} N_{J} J_{\xi}^{0} d \Delta  \tag{B4.8.4}\\
& \mathbf{M}_{I J}=\tilde{\mathbf{M}}_{I J}=\mathbf{I} \int_{\Omega_{0}} \rho_{0} N_{I} N_{J} d \Omega_{0} \tag{B4.8.5}
\end{align*}
$$

## Internal nodal force computation for element

1. $\mathbf{f}^{i n t}=0$
2. for all quadrature points $\xi_{Q}$
i. compute $\left[\mathcal{B}_{I j}^{0}\right]=\left[\partial N_{I}\left(\xi_{Q}\right) / \partial X_{j}\right]$ for all $I$
ii. $\mathbf{H}=\sum_{I} \mathbf{B}_{0 I} \mathbf{u}_{I} ; \quad H_{i j}=\frac{\partial N_{I}}{\partial X_{j}} u_{i I}$
iii. $\quad \mathbf{F}=\mathbf{I}+\mathbf{H}, \quad J=\operatorname{det}(\mathbf{F})$
iv. $\mathbf{E}=\frac{1}{2}\left(\mathbf{H}+\mathbf{H}^{T}+\mathbf{H}^{T} \mathbf{H}\right)$
v. if needed, compute $\dot{\mathbf{E}}=\Delta \mathbf{E} / \Delta t, \dot{\mathbf{F}}=\Delta \mathbf{F} / \Delta t, \quad \mathbf{D}=\operatorname{sym}\left(\dot{\mathbf{F}} \mathbf{F}^{\mathbf{1}}\right)$
vi. compute the PK2 stress $\mathbf{S}$ or Cauchy stress $\sigma$ by constitutive equation
vii. $\mathbf{P}=\mathbf{S F} \mathbf{F}^{T} \quad$ or $\mathbf{P}=J \mathbf{F}^{-1} \boldsymbol{\sigma}$
viii. $\mathbf{f}_{I}^{\text {int }} \leftarrow \mathbf{f}_{I}^{\text {int }}+\mathcal{B}_{O I}^{T} \mathbf{P} J_{\xi}^{0} \bar{w}_{Q}$ for all nodes I
end loop
$\bar{w}_{Q}$ are quadrature weights

Usually the shape functions are expressed in terms of element coordinates $\xi$, such as the area coordinates in triangular elements or reference coordinates in isoparametric elements. The derivatives with respect to the material coordinates are then found by

$$
\begin{equation*}
\mathbf{N}_{I, \mathbf{X}} \equiv \mathcal{B}_{I}^{0}=\mathbf{N}_{\xi} \mathbf{X}_{\xi}^{-1}=\mathbf{N}_{, \xi}\left(\mathbf{F}_{\xi}^{0}\right)^{-1} \tag{4.9.18}
\end{equation*}
$$

where $\mathbf{F}_{\xi}^{0}$ is the Jacobian between the material and intrinsic coordinates. As shown in Box 4.7, the Green strain tensor is usually not computed in terms of the deformation gradient $\mathbf{F}$, because the resulting computation is susceptible to round-off errors for small strains. Therefore the procedure shown in Eqs. (B4.7.6-7) is used. The total Lagrangian formulation can easily be adapted to constitutive equations expressed in terms of the Cauchy stress: it is only necessary to introduce the transormations shown in steps 2.vi-vii.

Voigt Form. It is of little use to write the nodal forces in terms of $\mathbf{P}$ using Voigt notation since $\mathbf{P}$ is not symmetric. Therefore, we will write the Voigt form in terms of the PK2 stress $\mathbf{S}$. Using the transformation $\mathbf{P}=\mathbf{S} \cdot \mathbf{F}^{T}$, the expression for the internal nodal forces becomes

$$
\begin{equation*}
f_{j l}^{i n t}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{i}} F_{j k} S_{i k} d \Omega_{0} \quad \text { or } \quad\left(\mathbf{f}_{I}^{i n t}\right)^{T}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial \mathbf{X}} \mathbf{S F}^{T} d \Omega_{0} \tag{4.9.19}
\end{equation*}
$$

We define a $\mathbf{B}_{0}$ matrix by

$$
\begin{equation*}
B_{i k l j}^{0}=\operatorname{sym}_{(i, k)}\left(\frac{\partial N_{I}}{\partial X_{i}} F_{j k}\right) \tag{4.9.20}
\end{equation*}
$$

Note that the above specializes to the updated form (4.5.21) where the current configuration and the reference configuration coincide, so that $F_{i j} \rightarrow \delta_{i j}$. The Voigt form of this matrix (see Appendix A) is

$$
\begin{array}{ll}
B_{i k l j}^{0} \rightarrow B_{a b}^{0} & (i, k) \rightarrow a \text { by the Voigt kinematic rule }  \tag{4.9.21}\\
& (I, j) \rightarrow b \text { by the rectangular to column matrix rule }
\end{array}
$$

Similarly, $S_{i k}$ is converted to $S_{b}$ by the kinetic Voigt rule. Then

$$
\begin{equation*}
f_{a}^{\text {int }}=\int_{\Omega_{0}}\left(B_{a b}^{0}\right)^{T} S_{b} d \Omega_{0} \quad \text { or } \quad \mathbf{f}=\int_{\Omega_{0}} \mathbf{B}_{0}^{T}\{\mathbf{S}\} d \Omega_{0} \quad \text { or } \quad \mathbf{f}_{I}=\int_{\Omega_{0}} \mathbf{B}_{0 I}^{T}\{\mathbf{S}\} d \Omega_{0} \tag{4.9.22}
\end{equation*}
$$

The construction of the $\mathbf{B}_{0}$ matrix hinges on the correspondence between the index $a$ and the $i$ and $k$ indices given in Table A.?. Using this correspondence for a two-dimensional element, we obtain:

$$
\begin{align*}
B_{0 i k l j} \rightarrow & B_{0 a l j} \\
& i=1, k=1 \rightarrow a=1\left[B_{0 a j}\right]_{I}=\frac{\partial N_{I}}{\partial X} F_{j 1}=\frac{\partial N_{I}}{\partial X} \frac{\partial x_{j}}{\partial X} \\
& i=2, k=2 \rightarrow a=2\left[B_{0 a j}\right]_{I}=\frac{\partial N_{I}}{\partial Y} F_{j 2}=\frac{\partial N_{I}}{\partial Y} \frac{\partial x_{j}}{\partial Y}  \tag{4.9.23}\\
& i=1, k=2 \rightarrow a=3\left[B_{0 a j}\right]_{I}=\frac{\partial N_{I}}{\partial X} F_{j 2}+\frac{\partial N_{I}}{\partial Y} F_{j 1}=\frac{\partial N_{I}}{\partial X} \frac{\partial x_{j}}{\partial Y}+\frac{\partial N_{I}}{\partial Y} \frac{\partial x_{j}}{\partial X}
\end{align*}
$$

The $\mathbf{B}_{0 I}$ matrix is then written out by letting $j=1$ and 2 correspond to columns 1 and 2 of the matrix, respectively:

$$
\mathbf{B}_{0 I}=\left[\begin{array}{cc}
\frac{\partial N_{I}}{\partial X} \frac{\partial x}{\partial X} & \frac{\partial N_{I}}{\partial X} \frac{\partial y}{\partial X}  \tag{4.9.24}\\
\frac{\partial N_{I}}{\partial Y} \frac{\partial x}{\partial Y} & \frac{\partial N_{I}}{\partial Y} \frac{\partial y}{\partial Y} \\
\frac{\partial N_{I}}{\partial X} \frac{\partial x}{\partial Y}+\frac{\partial N_{I}}{\partial Y} \frac{\partial x}{\partial X} & \frac{\partial N_{I}}{\partial X} \frac{\partial y}{\partial Y}+\frac{\partial N_{I}}{\partial Y} \frac{\partial y}{\partial X}
\end{array}\right]
$$

In three dimensions, a similar procedure yields

$$
\left\{\begin{array}{ccc} 
& \frac{\partial N_{I}}{\partial X} \frac{\partial x}{\partial X} & \frac{\partial N_{I}}{\partial X} \frac{\partial y}{\partial X}  \tag{4.9.25}\\
\mathbf{B}_{0 I} & \frac{\partial N_{I}}{\partial Y} \frac{\partial x}{\partial Y} & \frac{\partial N_{I}}{\partial Y} \frac{\partial y}{\partial Y}
\end{array}\right.
$$

Many writers construct the $\mathbf{B}_{0 I}$ matrix through a sequence of multiplications by Boolean matrices. The procedure shown here can easily be coded and is much faster.

It can be easily shown that $\mathbf{B}_{0}$ relates the rate of Green strain $\dot{\mathbf{E}}$ to the node velocities by

$$
\begin{equation*}
\{\dot{\mathbf{E}}\}=\mathbf{B}_{0 I} \mathbf{v}_{I}=\mathbf{B}_{0} \dot{\mathbf{d}} \tag{4.9.27}
\end{equation*}
$$

The reader should be cautioned about one characteristic of the $\mathbf{B}_{0}$ matrix: although it carries a subscript nought, the matrix $\mathbf{B}_{0}$ is not time invariant. This can easily be seen from Eqs. (4.9.20) or (4.9.24-25), which show that the $\mathbf{B}_{0}$ matrix depends on $\mathbf{F}$, which varies with time.

The total Lagrangian equation for internal nodal forces, (4.9.22) can easily be reduced to the updated Lagrangian form, Eq. (4.5.14) without any transformations. This is accomplished by letting the configuration at a fixed time $t$ be the reference configuration. We then use the total Lagrangian formulation with this new reference configuration. It is immediately clear that

$$
\begin{equation*}
\mathbf{F}=\mathbf{I} \quad \text { or } \quad F_{i j}=\frac{\partial x_{i}}{\partial X_{j}}=\delta_{i j} \tag{4.9.28}
\end{equation*}
$$

since the two coordinates systems are now coincident at time $t$. There are several consequences of this:

$$
\begin{equation*}
\mathbf{B}_{0}=\mathbf{B} \quad \mathbf{S}=\sigma \quad \Omega_{0}=\Omega \quad \mathrm{J}=1 \quad d \Omega_{0}=d \Omega \tag{4.9.29}
\end{equation*}
$$

to verify this, compare (4.9.20) and (??); from Box 3.2 it follows that since $\mathbf{F}=\mathbf{I}$ and $\mathbf{S}=\sigma$. Then Eq. (4.9.22) becomes

$$
\begin{equation*}
\mathbf{f}_{I}=\int_{\Omega} \mathbf{B}_{I}^{T}\{\sigma\} d \Omega \tag{4.9.30}
\end{equation*}
$$

which agrees with Eq. (4.5.14). This process of instantenously making the current configuration the reference configuration is a helpful trick which we will again use later.

Example 4.8. Rod in Two Dimensions. Develop the internal nodal forces for a two-node rod element in two-dimensions. The bar element is shown in Fig. 4.12. It is in a uniaxial state of stress with the only nonzero stress along the axis of the bar.


Fig. 4.12. Rod element in rwo dimensions in total Lagrangian formulation
To simplify the formulation, we place the material coordinate system so that the $X$-axis coincides with the axis of the rod, as shown in Fig. 4.12, with the origin of the material coordinates at node 1 . The parent element coordinate is $\xi, \xi \in[0,1]$. The material coordinates are then related to the element coordinates by

$$
\begin{equation*}
X=X_{2} \xi=\ell_{0} \xi \tag{E4.8.1}
\end{equation*}
$$

where $\ell_{0}$ is the initial length of the element. In this example, the coordinates $X, Y$ are used in a somewhat different sense than before: it is no longer true that $\mathbf{x}(t=0)=\mathbf{X}$. However, the definition used here corresponds to a rotation and translation of $\mathbf{x}(t=0)$. Since neither rotation nor translation effects $\mathbf{E}$ or any strain measure, this choice of an $X, Y$ coordinate system is perfectly acceptable. We could have used the element coordinates $\xi$ as material coordinates, but this complicates the definition of physical strain components.

The spatial coordinates are given in terms of the element coordinates by

$$
\begin{align*}
& x=x_{1}(1-\xi)+x_{2} \xi  \tag{E4.8.2}\\
& y=y_{1}(1-\xi)+y_{2} \xi
\end{align*} \quad \text { or } \quad\left\{\begin{array}{l}
x \\
y
\end{array}\right\}=\left[\begin{array}{ll}
x_{1} & x_{2} \\
y_{1} & y_{2}
\end{array}\right]\left\{\begin{array}{c}
1-\xi \\
\xi
\end{array}\right\}
$$

or

$$
\begin{equation*}
\mathbf{x}(\xi, t)=\mathbf{x}_{I}(t) N_{I}(\xi) \tag{E4.8.3}
\end{equation*}
$$

where

$$
\left\{N_{I}(\xi)\right\}^{T}=\left[\begin{array}{ll}
(1-\xi) & \xi
\end{array}\right]=\left[\begin{array}{ll}
1-\frac{X}{\ell_{0}} & \frac{X}{\ell_{0}} \tag{E4.8.4}
\end{array}\right]
$$

The $B_{0}$ matrix as defined in (4.9.7) is given by

$$
\left[\mathcal{B}_{0 i I}\right] \equiv\left[\partial N_{I} / \partial X_{i}\right]^{T}=\left[\begin{array}{ll}
\frac{\partial N_{1}}{\partial X} & \frac{\partial N_{2}}{\partial X}
\end{array}\right]=\frac{1}{\ell_{0}}\left[\begin{array}{ll}
-1 & +1 \tag{E4.8.5}
\end{array}\right]
$$

where Eq. (4.8.1) has been used to give $\frac{\partial N_{I}}{\partial X}=\frac{1}{\ell_{0}} \frac{\partial N_{I}}{\partial \xi}$. The deformation gradient is given by (4.9.7):

$$
\mathbf{F}=\mathbf{x}_{I}\left(\mathrm{~B}_{I}^{0}\right)^{T}=\left[\begin{array}{ll}
x_{1} & x_{2}  \tag{E4.8.6}\\
y_{1} & y_{2}
\end{array}\right] \frac{1}{\ell_{0}}\left\{\begin{array}{c}
-1 \\
1
\end{array}\right\}=\frac{1}{\ell_{0}}\left[\begin{array}{ll}
x_{2}-x_{1} & y_{2}-y_{1}
\end{array}\right] \equiv \frac{1}{\ell_{0}}\left[\begin{array}{ll}
x_{21} & y_{21}
\end{array}\right]
$$

The deformation gradient $\mathbf{F}$ is not a square matrix for the rod since there are two space dimensions but only one independent variable describes the motion, (E4.8.2).

The only nonzero stress is along the axis of the rod. To take advantage of this, we use the nodal force formula in terms of the PK2 stress, since $S_{11}$ is the only nonzero component of this stress. For the nominal stress, $P_{11}$ is not the only nonzero component. The $X$ axis as defined here is corotational with the axis of the rod, so $S_{11}$ is always the stress component along the axis of the rod. Substituting (E4.8.5) and (E4.8.6) into Eq. (4.9.19) then gives the following expression for the internal nodal forces:

$$
\mathbf{f}_{\text {int }}^{T}=\int_{\Omega_{0}} \mathrm{~B}_{0}^{T} \mathbf{S} \mathbf{F}^{T} d \Omega_{0}=\int_{\Omega_{0}} N,_{\mathbf{X}} \mathbf{S F}^{T} d \Omega_{0}=\int_{\Omega_{0}} \frac{1}{\ell_{0}}\left[\begin{array}{c}
-1  \tag{E4.8.7}\\
+1
\end{array}\right]\left[\begin{array}{ll}
S_{11}
\end{array}\right] \frac{1}{\ell_{0}}\left[\begin{array}{ll}
x_{21} & y_{21}
\end{array}\right] d \Omega_{0}
$$

Since the deformation is constant in the element, we can assume the integrand is constant, so multiplying the integrand by the volume $A_{0} \ell_{0}$ we have

$$
\left[\begin{array}{cc}
f_{1 x} & f_{1 y}  \tag{E4.8.9}\\
f_{2 x} & f_{2 y}
\end{array}\right]^{\text {int }}=\frac{A_{0} S_{11}}{\ell_{0}}\left[\begin{array}{cc}
-x_{21} & -y_{21} \\
x_{21} & y_{21}
\end{array}\right]
$$

This result can be transformed to the result for the corotational formulation if we use Eq. (E3.9.8) and note that $\cos \theta=\frac{x_{21}}{\ell}$ and $\sin \theta=\frac{y_{21}}{\ell}$.

In Voigt notation, the nonzero entries of the $\mathbf{B}_{0}$ matrix are the first row of (4.9.24), so

$$
\mathbf{B}_{0 I}=\left[\begin{array}{ll}
x_{, X} N_{I, X} & y_{, X} N_{I, X}
\end{array}\right]=\left[\begin{array}{ll}
\cos \theta N_{I, X} & \sin \theta N_{I, X}
\end{array}\right]
$$

Noting that $N_{1, X}=-1 / \ell_{0}, N_{2, X}=1 / \ell_{0}$, we have that

$$
\mathbf{B}_{0}=\left[\begin{array}{ll}
\mathbf{B}_{1}^{0} & \mathbf{B}_{2}^{0}
\end{array}\right]=\frac{1}{\ell_{0}}\left[\begin{array}{llll}
-\cos \theta & -\sin \theta & \cos \theta & \sin \theta
\end{array}\right]
$$

The expression for the nodal forces, (4.5.19) then becomes

$$
\mathbf{f}^{\text {int }} \equiv\left\{\begin{array}{l}
f_{x 1} \\
f_{y 1} \\
f_{x 2} \\
f_{y 2}
\end{array}\right\}^{\text {int }}=\int_{\Omega_{0}} \mathbf{B}_{0}^{T}\{\mathbf{S}\} d \Omega_{0}=\int_{\Omega_{0}} \frac{1}{\ell_{0}}\left\{\begin{array}{c}
-\cos \theta \\
-\sin \theta \\
\cos \theta \\
\sin \theta
\end{array}\right\}\left\{S_{11}\right\} d \Omega_{0}
$$

Example 4.9. Triangular Element. Develop expressions for the deformation gradient, nodal internal forces and nodal external forces for the 3-node, linear displacement triangle. The element was developed in the updated Lagrangian formulation in Example 4.1; the element is shown in Fig. 4.2.

The motion of the element is given by the same linear map as in Example 4.1, Eq. (E4.1.2) in terms of the triangular coordinates $\xi_{I}$. The $\mathcal{B}_{0}$ matrix is given by (4.9.7):

$$
\left.\begin{array}{rl}
\mathcal{B}_{0 I} & =\left[\mathcal{B}_{j I}^{0}\right]=\left[\partial N_{I} / \partial X_{j}\right.
\end{array}\right], \left.\quad \mathcal{B}_{0}=\left[\begin{array}{lll}
\mathcal{B}_{01} & \mathcal{B}_{02} & \mathcal{B}_{03}
\end{array}\right]=\left[\begin{array}{lll}
\frac{\partial N_{1}}{\partial X} & \frac{\partial N_{2}}{\partial X} & \frac{\partial N_{3}}{\partial X} \\
\frac{\partial N_{1}}{\partial Y} & \frac{\partial N_{2}}{\partial Y} & \frac{\partial N_{3}}{\partial X}
\end{array}\right] \right\rvert\, \begin{array}{lll}
Y_{23} & Y_{31} & Y_{12}  \tag{E4.9.1}\\
& =\frac{1}{2 A_{0}}\left[\begin{array}{lll}
X_{32} & X_{13} & X_{21}
\end{array}\right] \\
A_{0} & =\frac{1}{2}\left(X_{32} Y_{12}-X_{12} Y_{32}\right)
\end{array}
$$

where $A_{0}$ is the area of the undeformed element and $X_{I J}=X_{I}-X_{J}, Y_{I J}=Y_{I}-Y_{J}$. These equations are identical to those given in the updated Lagrangian formulation except that the initial nodal coordinates and initial area are used. The internal forces are then given by (4.9.11b):

$$
\begin{align*}
\mathbf{f}_{\text {int }}^{T} & =\left[f_{i I}\right]=\left[\begin{array}{ll}
f_{1 x} & f_{1 y} \\
f_{2 x} & f_{2 y} \\
f_{3 x} & f_{3 y}
\end{array}\right]^{\text {int }}=\int_{\Omega_{0}} \mathcal{B}_{0}^{T} \mathbf{P} d \Omega_{0} \\
& =\int_{A_{0}} \frac{1}{2 A_{0}}\left[\begin{array}{ll}
Y_{23} & X_{32} \\
Y_{31} & X_{13} \\
Y_{12} & X_{21}
\end{array}\right]\left[\begin{array}{ll}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{array}\right] a_{0} d A_{0}=\frac{a_{0}}{2}\left[\begin{array}{ll}
Y_{23} & X_{32} \\
Y_{31} & X_{13} \\
Y_{12} & X_{21}
\end{array}\right]\left[\begin{array}{ll}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{array}\right] \tag{E4.9.2}
\end{align*}
$$

Voigt Notation. The expression for the internal nodal forces in Voigt notation requires the $\mathbf{B}_{0}$ matrix. Using Eq. (4.9.24) and the derivatives of the shape functions in Eq. (E4.9.1) gives

$$
\mathbf{B}_{0}=\left[\begin{array}{cccccc}
Y_{23} x_{X} & Y_{23} y_{\prime_{X}} & Y_{31} x_{X} & Y_{31} y_{X} & Y_{12} x_{\prime_{X}} & Y_{12} y_{X}  \tag{E4.9.3}\\
X_{32} x_{Y} & X_{32} y_{\prime_{Y}} & X_{13} x_{Y} & X_{13} y_{\gamma_{Y}} & X_{21} x_{Y} & X_{21} y_{Y} \\
\left\lfloor Y_{23} x_{\prime_{Y}}+X_{32} x_{X}\right. & Y_{23} y_{\prime_{Y}}+X_{32} y_{X X} & Y_{31} x_{Y}+X_{13} x_{X} & Y_{31} y_{\prime_{Y}+X_{13} y_{X}} & Y_{12} x_{Y}+X_{21} x_{X} & Y_{12} y_{Y}+X_{21} y_{X}
\end{array}\right\rfloor
$$

The terms of the $\mathbf{F}$ matrix, $x_{x_{X}}, y_{y_{X}}$, etc., are evaluated by Eq. (4.9.6); for example:

$$
\begin{equation*}
x_{X}=N_{I, X} x_{I}=\frac{1}{2 A_{0}}\left(Y_{23} x_{1}+Y_{31} x_{2}+Y_{12} x_{3}\right) \tag{E4.9.4}
\end{equation*}
$$

Note that the $\mathbf{F}$ matrix is constant in the element, and so is $\mathbf{B}_{0}$. The nodal forces are then given by Eq. (4.9.22):

$$
\mathbf{f}^{i n t}=\left\{f_{a}\right\}=\left\{\begin{array}{l}
f_{1 x}  \tag{E4.9.5}\\
f_{1 y} \\
f_{2 x} \\
f_{2 y} \\
f_{3 x} \\
f_{3 y}
\end{array}\right\}^{i n t}=\int_{\Omega_{0}} \mathbf{B}_{0}^{T}\left\{\begin{array}{l}
S_{11} \\
S_{22} \\
S_{12}
\end{array}\right\} d \Omega_{0}
$$

Example 4.10. Two-Dimensional Isoparametric Element. Construct the discrete equations for two- and three-dimensional isoparametric elements in indicial matrix notation and Voigt notation. The element is shown in Fig. 4.4; the same element in the updated Lagrangian form was considered in Example 4.2.

The motion of the element is given in Eq. (E4.2.1), followed by the shape functions and their derivatives with respect to the spatial coordinates. The key difference in the formulation of the isoparametric element in the total Lagrangian formulation is that the matrix of derivatives of the shape functions with respect to the material coordinates must be found. By implicit differentiation

$$
\left\{\begin{array}{l}
N_{I, X}  \tag{E4.10.1}\\
N_{I, Y}
\end{array}\right\}=\mathbf{X}_{, \xi}^{-1}\left\{\begin{array}{l}
N_{I \xi} \\
N_{I, \eta}
\end{array}\right\}=\left(\mathbf{F}_{\xi}^{0}\right)^{-1}\left\{\begin{array}{l}
N_{I, \xi} \\
N_{I, \eta}
\end{array}\right\}
$$

where

$$
\begin{equation*}
\mathbf{X}_{, \xi}=\mathbf{X}_{I} N_{I, \xi} \quad \text { or } \quad \frac{\partial X_{i}}{\partial \xi_{j}}=X_{i I} \frac{\partial N_{I}}{\partial \xi_{j}} \tag{E4.10.2}
\end{equation*}
$$

Writing out the above gives

$$
\left[\begin{array}{ll}
X_{, \xi} & X_{, \eta}  \tag{E4.10.3}\\
Y_{\zeta}^{,} & Y_{\eta}
\end{array}\right]=\left\{\begin{array}{c}
X_{I} \\
Y_{I}
\end{array}\right\}\left[\begin{array}{ll}
N_{I, \xi} & N_{I, \eta}
\end{array}\right]
$$

which can be evaluated from the shape functions and nodal coordinates; details are given for the 4node quadrilateral in Eqs. (E4.2.7-8) in terms of the updated coordinates and the formulas for the material coordinates can be obtained by replacing $\left(x_{I}, y_{I}\right)$ by $\left(X_{I}, Y_{I}\right)$. The inverse of $\mathbf{X}_{, \xi}$ is then given by

$$
\mathbf{X}_{, \xi}^{-1}=\left[\begin{array}{cc}
X_{, \xi} & X_{,} \\
Y, \xi & Y_{\eta}^{\prime}
\end{array}\right]^{-1}=\frac{1}{J_{0}^{\xi}}\left[\begin{array}{cc}
Y_{,} & -X_{, \eta} \\
-Y_{, \xi} & X_{, \xi}
\end{array}\right]^{-1}=\left[\begin{array}{cc}
\xi_{, X} & \eta_{, X} \\
\xi_{, Y} & \eta_{, Y}
\end{array}\right]
$$

where the determinant of the Jacobian between the parent and reference configurations is given by

$$
J_{0}^{\xi}=X_{, \xi} Y_{\eta}-Y_{\zeta} X_{\eta}
$$

The $\mathcal{Z}_{0 I}$ matrices are given by

$$
\mathcal{B}_{0 I}^{T}=\left[\begin{array}{ll}
N_{I, X} & N_{I, Y}
\end{array}\right]=\left[\begin{array}{ll}
N_{I, \xi} & N_{I, \eta}
\end{array}\right] \mathbf{X}_{, \xi}^{-1}=\left[\begin{array}{ll}
N_{I, \xi} & N_{I, \eta}
\end{array}\right]\left[\begin{array}{ll}
\xi_{, X} & \eta_{, X}  \tag{E4.10.4}\\
\xi_{, Y} & \eta_{, Y}
\end{array}\right]
$$

The gradient of the displacement field $\mathbf{H}$ is given by

$$
\mathbf{H}=\mathbf{u}_{I} \mathcal{B}_{0 I}^{T}=\left\{\begin{array}{l}
\mathbf{u}_{x I}  \tag{E4.10.5}\\
\mathbf{u}_{y I}
\end{array}\right\}\left[\begin{array}{ll}
N_{I, X} & N_{I, Y}
\end{array}\right]
$$

The deformation gradient is then given by

$$
\begin{equation*}
\mathbf{F}=\mathbf{I}+\mathbf{H} \tag{E4.10.6}
\end{equation*}
$$

The Green strain $\mathbf{E}$ is obtained from (B4.7.4) and the the stress $\mathbf{S}$ is evaluated by the constitutive equation; the nominal stress $\mathbf{P}$ can then be computed by $\mathbf{P}=\mathbf{S F}{ }^{T}$; see Box 3.2.

The internal nodal forces are given by Eq. (4.9.11b):

$$
\left(\mathbf{f}_{I}^{\text {int }}\right)^{T}=\int_{\Omega_{0}} \mathcal{B}_{0 I}^{T} \mathbf{P} d \Omega_{0}=\int_{-1-1}^{1} \int_{I-1}^{1}\left[\begin{array}{ll}
N_{I, X} & N_{I, Y}
\end{array}\right]\left[\begin{array}{ll}
P_{11} & P_{12}  \tag{E4.10.7}\\
P_{21} & P_{22}
\end{array}\right] J_{0}^{\xi} d \xi d \eta
$$

where

$$
\begin{equation*}
J_{0}^{\xi}=\operatorname{det}\left(\mathbf{X}_{\xi}\right)=\operatorname{det}\left(\mathbf{F}_{\xi}^{0}\right) \tag{E4.10.8}
\end{equation*}
$$

If the Voigt form is used, the internal forces are computed by Eq. (4.9.22) in terms of $\mathbf{S}$. The external nodal forces, particularly those due to pressure, are usually best computed in the updated form. The mass matrix was computed in the total Lagrangian form in Example 4.2.

Example 4.12. Three-Dimensional Element. Develop the strain and nodal force equations for a general three-dimensional element in the total Lagrangian format. The element is shown in Figure 4.5. The parent element coordinates are $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right) \equiv(\xi, \eta, \zeta)$ for an isoparametric element, $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right)$ for a tetrahedral element, where for the latter $\xi_{i}$ are the volume (barycentric) coordinates.

Matrix Form. The standard expressions for the motion, Eqs. (4.9.1-5) are used. The deformation gradient is given by Eq. (4.9.6). The Jacobian matrix relating the reference configuration to the parent is

$$
\mathbf{X}, \xi=\left[\begin{array}{ccc}
X, \xi & X_{\eta} & X_{, \xi} \\
Y, \xi & Y, \eta & Y_{, \xi} \\
Z, \xi & Z, \eta & Z, \xi
\end{array}\right]=\mathbf{X}_{I} \mathbf{B}_{0 I}^{T}=\left\{X_{I}\right\}\left[\partial N_{I} / \partial \xi_{j}\right]=\left\{\begin{array}{l}
X_{I} \\
Y_{I} \\
Z_{I}
\end{array}\right\}\left[\begin{array}{lll}
N_{I, \xi} & N_{I, \eta} & N_{I, \xi}
\end{array}\right] \text { (E4.12.1) }
$$

The deformation gradient is given by

$$
\left[F_{i j}\right]=\left[x_{i I}\right]\left[\frac{\partial N_{I}}{\partial X_{J}}\right]=\left[\begin{array}{l}
x_{1}, \ldots, x_{N}  \tag{E4.12.2}\\
y_{1}, \ldots, y_{N} \\
z_{1}, \ldots, z_{N}
\end{array}\right]\left\{\begin{array}{l}
N_{I, X} \\
N_{I, Y} \\
N_{I, Z}
\end{array}\right\}
$$

where

$$
\left\{\frac{\partial N_{I}}{\partial X_{j}}\right\}=\left\{\begin{array}{l}
N_{I, X}  \tag{E4.12.3}\\
N_{I, Y} \\
N_{I, Z}
\end{array}\right\}=\left[\frac{\partial N_{I}}{\partial \xi_{k}}\right]\left[\frac{\partial \xi_{k}}{\partial X_{j}}\right]=\left[\frac{\partial N_{I}}{\partial \xi_{k}}\right] \mathbf{X}_{\xi}^{-1}
$$

where $\mathbf{X}, \xi_{\xi}^{-1}$ is evaluated numerically from Eq. (E4.12.1). The Green-strain tensor can be computed directly from $\mathbf{F}$, but to avoid round-off errors, it is better to compute

$$
\left[H_{i j}\right]=\left[u_{i I}\right]\left\{\frac{\partial N_{I}}{\partial X_{j}}\right\}=\left[\begin{array}{l}
u_{x 1}, \ldots, u_{x n}  \tag{E4.12.4}\\
u_{y 1}, \ldots, u_{y n} \\
u_{z 1}, \ldots, u_{z n}
\end{array}\right]\left\{\begin{array}{l}
\partial N_{I, X} \\
\partial N_{I, Y} \\
\partial N_{I, Z}
\end{array}\right\}
$$

The Green-strain tensor is then given by Eq. (???).
If the constitutive law relates the PK2 stress $\mathbf{S}$ to $\mathbf{E}$, the nominal stress is then computed by $\mathbf{P}=\mathbf{S} \mathbf{F}^{T}$, using $\mathbf{F}$ from Eq. (??.2). The nodal internal forces are then given by

$$
\left\{\begin{array}{l}
f_{x I}  \tag{E4.12.5}\\
f_{y I} \\
f_{z I}
\end{array}\right\}^{\text {int }}=\int_{\Delta}\left\{\begin{array}{l}
N_{I, X} \\
N_{I, Y} \\
N_{I, Z}
\end{array}\right\}=\left[\begin{array}{lll}
P_{11} & P_{12} & P_{13} \\
P_{21} & P_{22} & P_{23} \\
P_{31} & P_{32} & P_{33}
\end{array}\right\} J_{\xi}^{0} d \Delta
$$

where $J_{\xi}^{0}=\operatorname{det}(\mathbf{X}, \xi)$.
Voigt Form. All of the variables needed for the evaluation of the $\mathcal{B}_{0}$ matrix given in Eq. (???) can be obtained from Eq. (E???). In Voigt form

$$
\begin{align*}
& \{\mathbf{E}\}^{T}=\left[E_{11}, E_{22}, E_{33}, 2 E_{23}, 2 E_{13}, 2 E_{12}\right]  \tag{E4.12.6}\\
& \{\mathbf{S}\}^{T}=\left[S_{11}, S_{22}, S_{33}, S_{23}, S_{13}, S_{12}\right]
\end{align*}
$$

The rate of Green-strain can be computed by Eq. (???):

$$
\begin{align*}
& \{\dot{\mathbf{E}}\}=\mathbf{B}_{0} \dot{\mathbf{d}} \\
& \dot{\mathbf{d}}=\left[u_{x 1}, u_{y 1}, u_{z 1}, \ldots u_{x n}, u_{y n}, u_{z n}\right] \tag{E4.12.7}
\end{align*}
$$

The Green strain is computed by the procedure in Eq. (???). The nodal forces are given by

$$
\begin{equation*}
\mathbf{f}_{I}^{i n t}=\int_{\Delta} \mathrm{B}_{0 I}^{T}\{\mathbf{S}\} J_{0}^{\xi} d \Delta \tag{E4.12.8}
\end{equation*}
$$

4.9.3. Variational Principle. For static problems, weak forms for nonlinear analysis with path-independent materials can be obtained from variational principles. For many nonlinear problems, variational principles can not be formulated. However, when constitutive equations and loads are path-independent and nondissipative, a variational priniciple can be written because the stress and load can be obtained from potentials. The materials for which stress is derivable from a potential are called hyperelastic materials, see Section 5.4. In a hyperelastic material, the nominal stress is given in terms of a potential by Eq (5.4.113) which is rewritten here

$$
\begin{equation*}
\mathbf{P}^{T}=\frac{\partial w}{\partial \mathbf{F}}, \text { or } P_{j i}=\frac{\partial w}{\partial F_{i j}}, \text { where } w=\rho w^{i n t}, \quad \mathcal{W}^{i n t}=\int_{\Omega_{0}} w d \Omega_{0} \tag{4.9.28}
\end{equation*}
$$

Note the order of the subscripts on the stress, which follows from the definition.
For the existence of a variational principle, the loads must also be conservative, i.e. they must be independent of the deformation path. Such loads are also derivable from a potential, i.e. the loads must be related to a potential so that

$$
\begin{align*}
& \mathcal{W}^{e x t}(\mathbf{u})=\int_{\Omega_{0}} w_{b}^{e x t}(\mathbf{u}) d \Omega_{0}+\int_{\Gamma_{t}} w_{t}^{e x t}(\mathbf{u}) d \Gamma_{0} \\
& b_{i}=\frac{\partial w_{b}^{e x t}}{\partial u_{i}} \quad \bar{t}_{i}^{0}=\frac{\partial w_{t}^{e x t}}{\partial u_{i}} \tag{4.9.29b}
\end{align*}
$$

Theoem of Stationary Potential Energy. When the loads and constitutive equations posses potentials, then the stationary points of

$$
\begin{equation*}
\mathcal{W}(\mathbf{u})=\mathcal{W}^{i n t}(\mathbf{u})-\mathcal{W}^{e x t}(\mathbf{u}), \quad \mathbf{u}(\mathbf{X}, t) \in \mathcal{U} \tag{4.9.30}
\end{equation*}
$$

satisfies the strong form of the equilibrium equation (B4.5.2b). The equilibrium equation which emanates from this statienary principle is written in terms of the displacements by incorporating the constitutive equation and strain-displacement equation. This stationary principle applies only to static problems.

The theorem is proven by showing the equivalence of the stationary principle to the weak form for equilibrium, traction boundary conditions and the interior continuity conditions. We first write the stationary condition of (4.9.30), which gives

$$
\begin{equation*}
0=\delta \mathcal{W}(\mathbf{u})=\int_{\Omega_{0}}\left(\frac{\partial w}{\partial F_{i j}} \delta F_{i j} d \Omega-\frac{\partial w_{b}^{e x t}}{\partial u_{i}} \delta u_{i}\right) d \Omega_{0}-\int_{\Gamma_{0}} \frac{\partial w_{t}^{e x t}}{\partial u_{i}} \delta u_{i} d \Gamma_{0} \tag{4.9.31}
\end{equation*}
$$

Substituting Eqs. (4.9.28) and (4.9.29) into the above gives

$$
\begin{equation*}
0=\int_{\Omega_{0}}\left(P_{j i} \delta F_{i j}-\rho_{0} b_{i} \delta u_{i}\right) d \Omega_{0}-\int_{\Gamma_{0}} t_{i}^{0} \delta u_{i} d \Gamma_{0} \tag{4.9.32}
\end{equation*}
$$

which is the weak form given in Eq. (4.8.7) for the case when the accelerations vanish. The same steps given in Section 4.8 can then be used to establish the equivalence of Eq. (4.8.7) to the strong form of the equilibrium equation.

Stationary principles are thus in a sense more restrictive weak forms: they apply only to conservative, static problems. However they can improve our understanding of stability problems and are used in the study of the existence and uniqueness of solutions.

The discrete equations are obtained from the stationary principle by using the usual finite element approximation to motion with a Lagrangian mesh, Eqs. (4.12) to (4.9.5), which we write in the form

$$
\begin{equation*}
\mathbf{u}(\mathbf{X}, t)=\mathbf{N}(\mathbf{X}) \mathbf{d}(t) \tag{4.9.33}
\end{equation*}
$$

The potential energy can then be expressed in terms of the nodal displacements, giving

$$
\begin{equation*}
W(\mathbf{d})=W^{i n t}(\mathbf{d})-W^{\text {ext }}(\mathbf{d}) \tag{4.9.34}
\end{equation*}
$$

The solutions to the above correspond to the stationary points of this function, so the discrete eqautions are

$$
\begin{equation*}
0=\frac{\partial W(\mathbf{d})}{\partial \mathbf{d}}=\frac{\partial W^{i n t}(\mathbf{d})}{\partial \mathbf{d}}-\frac{\partial W^{e x t}(\mathbf{d})}{\partial \mathbf{d}} \equiv \mathbf{f}^{i n t}-\mathbf{f}^{e x t} \tag{4.9.35}
\end{equation*}
$$

It will be shown in Chapter 6, that when the equilibrium point is stable, the potential energy is a minimum.

Example 4.11. Rod Element by Stationary Principle. Consider a structural model consisting of two-node rod elements in three dimensions. Let the internal potential energy be given by

$$
\begin{equation*}
w=\frac{1}{2} C^{S E} E_{11}^{2} \tag{E4.11.1}
\end{equation*}
$$

and let the only load on the structure be gravity, for which the external potential is

$$
\begin{equation*}
w^{e x t}=-\rho_{0} g z \tag{E4.11.2}
\end{equation*}
$$

where $g$ is the acceleration of gravity. Find expressions for the internal and external nodal forces of an element.

From Eqs. (4.9.28) and (E4.11.1), the total internal potential is given by

$$
\begin{equation*}
w^{i n t}=\sum_{e} W_{e}^{i n t}, \quad W_{e}^{i n t}=\frac{1}{2} \int_{\Omega_{0}^{e}} C^{S E} E_{11}^{2} d \Omega_{0} \tag{E4.11.3}
\end{equation*}
$$

For the two-node element, the displacement field is linear and the Green strain is constant, so Eq. (E4.11.3) can be simplified by multiplying the integrand by the initial volume of the element $A_{0} \ell_{0}$ :

$$
\begin{equation*}
W_{e}^{i n t}=\frac{1}{2} A_{0} \ell_{0} C^{S E} E_{11}^{2} \tag{E4.11.4}
\end{equation*}
$$

To develop the internal nodal forces, we will need the derivatives of the Green strain with respect to the nodal displacements. Since the strain is constant in the element, Eq. (3.3.1) (also see Eq. (??)) gives:

$$
\begin{equation*}
E_{11}=\frac{\ell^{2}-\ell_{0}^{2}}{2 \ell_{0}^{2}}=\frac{\mathbf{x}_{21} \cdot \mathbf{x}_{21}-\mathbf{X}_{21} \cdot \mathbf{X}_{21}}{2 \ell_{0}^{2}} \tag{E4.11.5}
\end{equation*}
$$

where $\mathbf{x}_{I J} \equiv \mathbf{x}_{I}-\mathbf{x}_{J}, \mathbf{X}_{I J} \equiv \mathbf{X}_{I}-\mathbf{X}_{J}$. Noting that

$$
\begin{equation*}
\mathbf{x}_{I J} \equiv \mathbf{X}_{I J}+\mathbf{u}_{I J} \tag{E4.11.6}
\end{equation*}
$$

where $\mathbf{u}_{I J} \equiv \mathbf{u}_{I}-\mathbf{u}_{J}$ are the nodal displacements and substituting Eq. (E4.11.6) into Eq. (4.11.5) gives, after some algebra,

$$
\begin{equation*}
E_{11}=\frac{2 \mathbf{X}_{21} \cdot \mathbf{u}_{21}+\mathbf{u}_{21} \cdot \mathbf{u}_{21}}{2 \ell_{0}^{2}} \tag{E4.11.7}
\end{equation*}
$$

The derivatives of $E_{x}^{2}$ with respect to the nodal displacements are then given by

$$
\begin{equation*}
\frac{\partial\left(E_{x}^{2}\right)}{\partial \mathbf{u}_{2}}=\frac{\mathbf{X}_{21}+\mathbf{u}_{21}}{\ell_{0}^{2}}=\frac{\mathbf{x}_{21}}{\ell_{0}^{2}}, \quad \frac{\partial\left(E_{x}^{2}\right)}{\partial \mathbf{u}_{1}}=-\frac{\mathbf{X}_{21}+\mathbf{u}_{21}}{\ell_{0}^{2}}=-\frac{\mathbf{x}_{21}}{\ell_{0}^{2}} \tag{E4.11.8}
\end{equation*}
$$

Using the definition for internal nodal forces in conjunction with Eqs. (E4.11.4) and (E4.11.8) gives

$$
\begin{equation*}
\mathbf{f}_{2}^{i n t}=-\mathbf{f}_{1}^{\text {int }}=\frac{A_{0} C^{S E} E_{x} \mathbf{x}_{21}}{\ell_{0}} \tag{E4.11.9}
\end{equation*}
$$

By using the fact that $S_{x}=C^{S E} E_{x}$, it follows that

$$
\left(\mathbf{f}_{2}^{i n t}\right)^{T}=-\left(\mathbf{f}_{1}^{i n t}\right)^{T}=\frac{A_{0} S_{x}}{\ell_{0}}\left[\begin{array}{lll}
x_{21} & y_{21} & z_{21} \tag{E4.11.10}
\end{array}\right]
$$

This result, as expected, is identical to the result obtained for the bar by the principle of virtual work, Eq. (E4.8.9). The external potential for a gravity load is given by

$$
\begin{equation*}
\mathcal{W}^{e x t}=-\int_{\Omega_{0}} \rho_{0} g z d \Omega_{0} \tag{E4.11.11}
\end{equation*}
$$

The external potential is independent of $x$ or $y$, and $\mathcal{W}_{z}^{\text {ext }}=\mathcal{W}_{u_{z}}^{\text {ext }}$. If we make the finite element approximation $z=z_{I} N_{I}$, where $N_{I}$ are the shape functions given in Eq. (E4.8.4) then

$$
\begin{equation*}
\mathcal{W}^{e x t}=-\int_{\Omega_{0}} \rho_{0} g z_{I} N_{I} d \Omega_{0} \tag{E4.11.12}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{z I}^{e x t}=\frac{\partial \mathcal{W}^{e x t}}{\partial u_{z I}}=-\int_{0}^{1} \rho_{0} g z_{I} N_{I}(\xi) \ell_{0} A_{0} d \xi=-\frac{1}{2} A_{0} \ell_{0} \rho_{0} g \tag{E4.11.13}
\end{equation*}
$$

so the external nodal force on each node is half the force on the rod element due to gravity.

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